The Informativeness of Prices: Dispersed Asymmetric Information and Higher-Order Beliefs

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Abstract

This paper analyzes how higher-order expectations affect the informativeness of asset prices in a world with dispersed asymmetric information and a non-atomistic player. I show that higher-order expectations create a novel trade-off between peoples' knowledge about the driving forces of the economy and peoples' knowledge about the beliefs of others. Because of this trade-off, markets one would think should be close to "informationally efficiency" – those with many agents with precise private information – may be further away than previously thought. I demonstrate that more agents and more precise private information can decrease welfare for everyone by decreasing uncertainty about future prices – even for the agents who receive superior private information. To illustrate these effects, I develop a flexible solution technique for linear dynamic rational expectations models with dispersed asymmetric information and type-specific shocks affecting an endogenous variable.

1 Introduction

This paper asks how higher-order expectations affect the informativeness of prices. Couched within the context of an asset pricing model, I show that higher-order expectations are critical for understanding the "informational efficiency" of prices. Specifically, I demonstrate that propositions typically thought to decrease the informativeness of market-clearing prices – such as decreases in the number of players or the precision of their private information – may, in fact, be welfare enhancing once we account for higher-order beliefs.

Keynes (1936) famously likened investment to a beauty contest, where contestants are supposed to guess what the average opinion is. But as contestants recognize that everyone else is also forming an opinion about what the average opinion is likely to be, they are forced to guess what the average opinion expects of the average opinion – i.e. into a situation where they need to form even higher, higher-order expectations. This analogy aptly describes the outcome of a competitive market where traders have different, and possibly asymmetric, information about some underlying asset. Traders in this market

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will realize that the price not only reflects the underlying fundamentals but also the average expectation of those fundamentals, the average expectation of the average expectation of those fundamentals, and so on. To determine the most likely path for future prices, market participants will therefore need to form expectations of all higher-order beliefs, and recognize that those some beliefs partially reflect their own opinion. The aim of this paper is to analyze how uncertainty about these higher-order expectations affects the informativeness of market-clearing prices.

A large body of literature has, following Grossman (1978), studied the information content of prices in essentially static prediction games, where each player attempts to infer the value of some underlying fundamental.[1] Although a useful abstraction for several theoretical questions, it disregards by assumption the role played by uncertainty about higher-order expectations. Several papers have been devoted to dynamic extensions of the Grossman framework, incorporating higher-order beliefs.[2] But these have all focused solely on the consequences of higher-order thinking for the demand and dynamics of prices – and not on their implication for their information content.

This paper takes a first step towards making up for this neglect. I show that higher-order expectations have important implications for the "informational efficiency" of competitive prices. In particular, I argue that market-clearing prices are not necessarily a good aggregator of information if there are "many" players with precise private signals. This contrasts with the results in Hellwig (1980).

To do this, I first construct a simple prediction game, where players have to infer both the value of some underlying fundamental and – in the spirit of the beauty contest – what others believe about that fundamental. Each player in the prediction game has access two signals: a private signal about the underlying fundamental and an endogenous public signal that reflects the average beliefs in the population. This allows me in a simple framework to demonstrate the role of higher-order expectations for the informativeness of prices. I then construct a dynamic asset pricing model for a richer description of this relationship. In the asset pricing model, over-lapping generations of traders allocate their wealth between a risky and risk-free asset. The risky asset price is driven in part by an unobserved shock to asset supply. All traders in the model observe the price of the risky asset in addition to their own private signal about the supply shock. Compared to models used by, for instance, Singleton (1987) and Rondina and Walker (2012), I add two new features: I allow for (1) differences in the precision of private information across traders and (2) for a non-atomistic trader type. This allows me to directly vary the precision of a subset of traders' private information. It also allows me to control "the commonality" of that information – how many people receive the same private signal – by altering the proportion of the population comprised by the non-atomistic trader type.

To solve the model, I develop a new solution technique for linear rational expectation models with dispersed information. Extending the work of Nimark (2011), I show that a truncated state-space solution can be applied to a class of linear models with asymmetric dispersed information and idiosyncratic private shocks affecting endogenous variables. The framework is very flexible, and can be used to analyze many economic situations in which there is asymmetric information and learning from endogenous variables.[3] Importantly, it allows me to analyze the effects of higher-order expectations for the informativeness

[1] For recent papers along this line, see, for instance, Amador and Weill (2010) and the reference therein.

[2] See, for instance, Wang (1993), Allen et al. (2006) or Kasa et al (2013).

[3] See, for instance, Kohlhas (2014)

of market-clearing prices. Previous work has had to invoke “clever assumptions” to circumvent the infinite regress of expectations that happens in models with endogenous signals and dispersed information (Townsend 1983).[4] This effectively limits the analysis to simple prediction games about unobserved fundamentals. In contrast, this paper deals explicitly with the infinite formation of higher-order beliefs.

I show that what affects welfare in the asset pricing model is uncertainty about future prices; not uncertainty about the driving forces of the economy per se. In equilibrium, asset prices are a combination of both the underlying fundamental of the economy and (higher-order) expectations of that fundamental. This potentially creates a trade-off. Changes to the commonality or precision of information makes this trade-off apparent: Increases in the commonality of information or decreases in the precision of private information increase uncertainty about the fundamental but – importantly – decrease people’s uncertainty about others’ beliefs. Future prices can therefore become easier to predict, improving welfare, despite everyone knowing less about the supply fundamental affecting the price.

Increasing the commonality of information – by increasing, for instance, the relative size of a type in the market – makes the asset price reflect, to a greater extent, that type’s expectation, and less the independent private information of others. The skewed combination of independent private signals in the price decreases its informativeness – everyone becomes less sure about the value of the underlying fundamental.[5] This, all else equal, increases uncertainty about future prices. However, inferring higher-order beliefs also becomes easier: the larger type’s expectation now comprises a greater share of the average expectation, partially revealed to others through the observation of the market-clearing price. Additionally, it also becomes easier to infer the average expectation of that type’s expectation, and so on. Higher-order belief uncertainty decreases. This, in turn, decreases uncertainty about future prices for everyone, all else equal. The result of the combined impact of both effects is – for most parameter values – a u-shaped relationship between the commonality of information and uncertainty about next period’s price. For low values of the commonality of information, the increase in uncertainty about the fundamental dominates, increasing uncertainty about future prices and lowering welfare. For large values, however, it is the other way around.

Changes to the precision of private information about the fundamental likewise creates a trade-off between people’s knowledge of the underlying fundamental and knowledge of other people’s beliefs of that fundamental. On the one hand, poorer private information for a segment of the population implies that the price becomes a combination of worse signals about the fundamental, increasing uncertainty about future prices. On the other hand, however, decreased informational disparity, caused by poorer private information, causes people to be surer of each other’s expectations since everyone attaches a greater weight to the publicly observed price when forming their expectations. Eventually, this second effect can become so strong that it can offset the transmission of poorer private information through prices and increase welfare for everyone in the economy – despite people receiving poorer private information. Another u-shaped relationship emerges. This time between the precision of private information and

[4] These range from constructing a hierarchical information structure with limited dispersion of information; assuming information is only short-lived; keeping the payoff relevant variable constant over time; to last, assuming a finite environment in which the asset is liquidated at a certain date. For a thorough discussion of these see Makarov and Rhytzkov (2012)

[5] Say you have $x > 1$ equally precise independent normal signals about some fundamental. The best predictor of the fundamental that you can construct from these signals is the arithmetic average. Attaching more weight to one signal than the other will uniformly increase the mean-squared error of the forecast.

uncertainty about next period's price.

More broadly speaking, the two key comparative statics that I consider in this paper – changes in the commonality and the precision of private information – demonstrate that to assess the "informational efficiency" of markets it is important to assess the effects on higher-order beliefs. Certain policies may imply a trade-off between knowledge about the state of the economy and knowledge of other people's beliefs of the economy. And, for some parameter configurations, the latter may be the more important consideration.

1.1 Related Literature

The study of how prices communicate superior private information to people has a long history in economics, and is simply too vast to cite every worthy paper in the space of one article. That said, the papers most closely related are: Hayek (1945), Grossman (1978), Hellwig (1980), Foster and Viswanathan (1996), Vives (1997) and more recently Wong (2008) and Amador and Weill (2010, 2011). These all focus on how information diffuses through market-clearing prices, but are constrained by focusing on explicitly static models with a given degree of informational disparity. Exceptions are Walker (2007), Bernhardt et al. (2010) and Kasa et al. (2012), who analyze the transmission of information in special cases of the above dynamic setup. But to the best of my knowledge, the transmission of information through endogenous variables in the more general framework presented here, and especially the role of higher-order expectations in the information dissemination, has not previously been studied.

The solution technique developed in this paper directly truncates the infinite dimensional state-space vector. Townsend (1983), on the contrary, resorts to a truncation strategy, where the state of the economy is revealed to all agents with a two period lag. By making private information short-lived, Townsend (1983) can make use of standard solution techniques. Hellwig (2002) and Hellwig and Venkateswaran (2009) extend this solution technique to allow for perfect state revelation with a "long lag", and illustrate a computationally efficient algorithm to solve for the equilibrium. Recently, Nimark (2011) has proposed to instead directly truncate the infinite state-space, consisting of higher-order beliefs, rather than allowing for information to be perfectly revealed after a certain date. The two approaches can be shown to be mathematically isomorphic. The latter is, however, computationally more convenient. Both types of truncation strategy have at least two advantages: First, explicitly modeling higher-order expectations increases our understanding of the link between higher-order expectations and the dynamics of the endogenous variables. Second, since the model remains in state-space form, standard likelihood based estimation techniques can readily be applied. This makes it possible to empirically quantify the importance of dispersed information for the dynamics of endogenous variables.

I extend the Nimark (2011) framework to accommodate differences in the precision of private information. This allows me to analyze, for instance, how superior private information by one group of economic agents – say well-informed traders or the central bank – diffuses through market-clearing prices to the rest of the population. It also, perhaps less obviously, is a necessary extension to find the equilibrium of models where type-specific shocks affect endogenous variables. This is, for instance, the case in the many economically relevant situations where the assumption of zero mass of agents is untenable. Kohlhas (2014) provides a business cycle example.

Alternative solutions techniques to truncation have been proposed. Extending the frequency domain techniques originally suggested by Futia (1981), Kasa (2000), Walker (2007), Bernhardt et al. (2010), Makarov and Rytchkov (2012), Kasa et al. (2012), and Rondina and Walker (2012) detail analytical solutions to models with dispersed information. While the proposed method is analytically elegant, it cannot in general be transformed into state-space form. Empirical validation of the models is therefore complicated. Nevertheless the approach should be considered complementary to the method developed in this paper.

1.2 Organization

The rest of this paper is organized as follows: The next section presents a higher-order expectation prediction game used to demonstrate the key mechanisms. Section 3 describes the extended Singleton (1987) asset pricing model, while Section 4 discusses its equilibrium with the infinite state-space vector. Section 5 presents the truncated state-space solution, and illustrates a computationally convenient algorithm to find the market equilibrium. Section 6 discusses the implications of changes to the commonality and precision of private information for the informativeness of the market-clearing price, and hence welfare. Section 7 concludes. Technical details are relegated to Appendix A to E.

2 A Basic Model

I start by considering two examples that liken a financial market to a simple prediction game, where each person attempts to infer the value of some underlying fundamental but also, importantly, what others are thinking about that fundamental. The first of these examples varies the commonality of information – how many people receive the same information – keeping the precision of individual's private information constant; the second does the opposite. Using these two examples, I demonstrate the key insight of this paper: the potential for a trade-off between how much an endogenous public signal reveals about an underlying fundamental and how much that signal reveals about other people's expectation of the fundamental. If the latter matter sufficiently in the determination of asset prices – as the next section shows indeed can be the case – this demonstrates the possibility for a trade-off between how much people know about the fundamental driving the economy and people's knowledge of future asset prices.

2.1 The Commonality of Information

Suppose that there are two types of agents: a continuum of measure $1 - m$ of type x and a continuum of measure m of type z. The ex-ante utility of an individual agent of type x is given by:

$$u_x^i = \mathbb{E}\left[\theta - \mathbb{E}_x^i(\theta)\right]^2 + \gamma\mathbb{E}\left[\mathbb{E}_z(\theta) - \mathbb{E}_x^i(\mathbb{E}_z(\theta))\right]^2,\ i \in [0,\, 1-m],$$

where $\theta \sim N(0,\, 1/\tau)$ and $\gamma \geq 0$. $\mathbb{E}_x^i[\cdot]$ denotes person i's expectation of type x; $\mathbb{E}_z[\cdot]$ denotes the corresponding expectation for individuals of type z. The utility of type x individuals therefore depend on their ability to correctly guess the value of the underlying fundamental, θ, as well as type z agents beliefs of θ[6]. Individuals of type z, for simplicity, merely attempt to guess the value of the fundamental:

$$u_z^i = \mathbb{E}\left[\theta - \mathbb{E}_z(\theta)\right]^2, \; i \in]1-m, \, m].$$

To form their beliefs agents rely on a combination of private and public information. Their private information is summarized by the signals:

$$\begin{aligned} x^i \quad &= \quad \theta + \epsilon_x^i, \quad \epsilon_x^i \sim N(0, \, 1/\tau), \; \forall i \in [0, \, 1-m] \\ z \quad = z^j &= \quad \theta + \epsilon_z, \quad \epsilon_z \sim N(0, \, 1/\tau), \; \forall j \in]1-m, \, 1] \end{aligned}$$

where all the "noise terms" (ϵ_{xi} $\forall i$ and ϵ_z) are independent of θ and each other. τ denotes the precision of the two signals and $\int_0^{1-m} \epsilon_x^i di = 0$. I have above assumed that all agents of type z receive the *same* private signal. This allows me to control the commonality of information through the constant m – i.e. how disperse the information that people receive is – and hence illustrate the trade-off between knowing less about the fundamental but more about other people's expectation of that fundamental.

In addition to their private information, individuals of type x observe the public signal y_1:

$$y_1 = \int_0^m \mathbb{E}_z[\theta] \, di + \int_0^{1-m} \mathbb{E}_x^i[\theta] \, di. \tag{2.1}$$

The endogenous public signal, like the asset price in the following section, reflects the average expectation of people's beliefs. This is crucial for establishing the main trade-off.

To find the linear Bayesian equilibrium of the prediction game, I conjecture that the public signal is a linear function of the unobserved fundamental, θ, and the noise term in type z's private information, ϵ_z:

$$y_1 = k_0\theta + k_1\epsilon_z, \quad \{k_0, \, k_1\} \in \mathbb{R}^2. \tag{2.2}$$

Given this conjecture, x^i's expectation of θ equals:

$$\mathbb{E}_x^i[\theta] = w_x x_i + w_y \tilde{y}, \quad w_x \equiv \frac{\tau}{\left(2\tau + \left(\frac{k_0}{k_1}\right)^2 \tau\right)}, \; w_y \equiv \frac{\left(\frac{k_0}{k_1}\right)^2 \tau}{\left(2\tau + \left(\frac{k_0}{k_1}\right)^2 \tau\right)}$$

where $\tilde{y}_1 \equiv \theta + \left(\frac{k_1}{k_0}\right)\epsilon_z$. The average expectation of type x individuals is thus given by:

$$\bar{\mathbb{E}}_x[\theta] = w_x\theta + w_y\tilde{y}, \tag{2.3}$$

where $\bar{\mathbb{E}}_x[\cdot] \equiv \frac{1}{1-m}\int_0^{1-m} \mathbb{E}_x^i[\cdot] \, di$. Turning to type z agents, their common expectation of θ equals:

$$\mathbb{E}_z[\theta] = z \tag{2.4}$$

[6]Type z could here have been replaced with the average expectation across both types and all results would remain the same.

since I have, for simplicity, assumed that they only observe their private signal. Inserting (2.3) and (2.4) into (2.1) verifies our conjecture; we find that for a solution to exist:

$$\begin{aligned} k_0 &= m + (1-m)(w_x + w_y) && (2.5) \\ k_1 &= m + (1-m) w_y \left(\frac{k_1}{k_0} \right). && (2.6) \end{aligned}$$

The following proposition characterizes the equilibrium.

Proposition 1. *There is a unique equilibrium to the model, where:*

$$y_1 = k_0 \theta + k_1 \epsilon_z$$

and k_0 and k_1 solve (2.5) and (2.6), respectively. At the equilibrium solution, the informativeness of the public signal, y_1, about the fundamental, θ, is decreasing in the size of type z agents, m $\left(\frac{\partial k_1/k_0}{\partial m} > 0 \right)$.

Proof. See Appendix A. □

The intuition behind Proposition 1 is clear: Increasing the share of type z agents in the population increases the commonality of information. This decreases the extent to which the public signal combines different independent sources of information, decreasing its informativeness. All that people can learn from the public signal is the (weighted) average of individuals private information. The larger a fraction type z agents comprise of the total population, the more the public signal reflects just their noisy beliefs rather than the sum of individuals private information – the truly new bits of information that we could learn from the public signal. The "wisdom of the crowds", so to speak, decreases. This increases uncertainty about the underlying fundamental, θ, for everyone.

That said, uncertainty about the underlying fundamental is not the only aspect that matters for welfare. Uncertainty about other people's beliefs also enter the utility function. And, unlike with the fundamental, uncertainty about type z agents expectation of θ actually falls; the public signal is clearer indicator of their beliefs. This illustrates how variations in the size of market participants can imply offsetting effects on the predictability of future asset prices and hence welfare. One the one hand, people will know less about the underlying fundamentals that drive the asset price; on the other hand, however, they will know more about other people's beliefs about those fundamentals. Proposition 2 and Corollary 1 summarize these effects.

Proposition 2. *Agent x^i's uncertainty about the underlying fundamental, θ, equals:*

$$MSE_x^i(\theta) \equiv \mathbb{E}\left[\left(\theta - \mathbb{E}_x^i(\theta) \right)^2 \right] = \frac{1}{\tau} \frac{1}{2 + (k_0/k_1)^2}.$$

Uncertainty about the underlying fundamental is therefore an increasing function of the share of the population comprised by agents of type z, m $\left(\partial MSE_x^i(\theta) / \partial m > 0 \right)$. Uncertainty about type z agents' beliefs, however, is a decreasing function of m $\left(\partial MSE_x^i(\mathbb{E}_z \theta) / \partial m < 0 \right)$, where

$$MSE_x^i(\mathbb{E}_z \theta) = \frac{1}{\tau} \frac{(1 - k_1/k_0)^2}{1 + 2(k_1/k_0)^2}.$$

Proof. See Appendix A. □

Corollary 1. *There exists a value of $\gamma \in \mathbb{R}_+$ such that the ex-ante welfare of type x agents is decreasing in the mass of type z individuals, m.*

Proof. See Appendix A. □

The decrease in higher-order belief uncertainty, moreover, does not stop here: it also becomes easier to predict the "average expectation" and "cross-expectations" (type x of type z, for instance). This happens both because of the decrease in uncertainty about type z's beliefs and the compositional effect attaching more weight to its type. In fact, the entire infinite hierarchy that you can construct from "average" and "cross-expectations" of the fundamental becomes easier to forecast. If these higher-order expectations matter sufficiently in the determination of asset prices, despite knowing less about the underlying fundamentals, uncertainty about future prices can fall. This insight will be important later for the dynamic asset pricing model.

2.2 The Precision of Private Information and Higher-Order Beliefs

To demonstrate how varying the precision of private information also generates a trade-off between how much people know about an underlying fundamental and how much people know of others beliefs of that fundamental, I make two changes to the above model. First, I allow for variation in the precision of private information across types – specifically:

$$\begin{aligned} x^i &= \theta + \epsilon_x^i, \quad \epsilon_x^i \sim N\left(0,\, 1/\tau_x\right),\ \forall i \in [0,\, 1-m] \\ z = z^j &= \theta + \epsilon_z, \quad \epsilon_z \sim N\left(0,\, 1/\tau_z\right),\ \forall j \in]1-m,\, 1] \end{aligned}$$

where τ_x and τ_z now can differ and $\theta \sim N\left(0,\, 1/\tau_\theta\right)$. Second, I assume that each agent – both of type x and type z – now observes the new public signal y_2:

$$y_2 = \bar{\mathbb{E}}_x\left[\mathbb{E}_z\left(\theta\right)\right] + \epsilon_y, \quad \epsilon_y \sim N\left(0,\, 1/\tau_y\right), \tag{2.7}$$

where ϵ_y is assumed independent of θ, ϵ_{ix} $\forall i$ and ϵ_z; τ_y denotes the corresponding precision of the signal and $\bar{\mathbb{E}}_x\left[\cdot\right] \equiv \frac{1}{1-m}\int_0^{1-m} \mathbb{E}_x^i\left[\cdot\right] di$. Keynes (1936) famously described how asset prices not only reflect the underlying fundamental but also the average expectation of that fundamental, the average expectation of the average expectation of that fundamental, and so on. I have in (2.7) assumed that the public signal observed in the economy is a function of one of such higher-order beliefs. This allows me to demonstrate the role played by higher-order expectations in the dissemination of information through endogenous public signals.

To solve the model, I proceed as in the previous subsection: I conjecture that the public signal is a linear function of the unobserved fundamental, θ, and the public noise term, ϵ_y:

$$y_2 = k_0\theta + k_1\epsilon_y, \quad \{k_0,\, k_1\} \in \mathbb{R}^2. \tag{2.8}$$

Agent x^i's expectation of θ now equals:

$$\mathbb{E}_x^i\left[\theta\right] = w_x x_i + w_y^x \tilde{y}, \quad w_x \equiv \frac{\tau_x}{\left(\tau_\theta + \tau_x + \left(\frac{k_0}{k_1}\right)^2 \tau_y\right)}, \; w_y^x \equiv \frac{\left(\frac{k_0}{k_1}\right)^2 \tau_y}{\left(\tau_\theta + \tau_x + \left(\frac{k_0}{k_1}\right)^2 \tau_y\right)}$$

where $\tilde{y}_2 \equiv \theta + \left(\frac{k_1}{k_0}\right)\epsilon$. Similarly, to find type z agents' expectation of θ, I use the conjecture in (2.8) to find that:

$$\mathbb{E}_z\left[\theta\right] = w_z z + w_y^z \tilde{y}_2, \quad w_z \equiv \frac{\tau_z}{\left(\tau_\theta + \tau_z + \left(\frac{k_0}{k_1}\right)^2 \tau_y\right)}, \; w_y^z \equiv \frac{\left(\frac{k_0}{k_1}\right)^2 \tau_y}{\left(\tau_\theta + \tau_z + \left(\frac{k_0}{k_1}\right)^2 \tau_y\right)}. \tag{2.9}$$

The average expectation across type x individuals of type z's belief – the endogenous term in (2.7) – therefore equals:

$$\bar{\mathbb{E}}_x\left[\mathbb{E}_z\left[\theta\right]\right] = w_z\left(w_x \theta + w_y^x \tilde{y}\right) + w_y^z \tilde{y}. \tag{2.10}$$

Inserting (2.10) into (2.7) verifies our conjecture. For a solution to exist, the coefficients k_0 and k_1 have to satisfy:

$$k_0 = w_z\left(w_x + w_y^x\right) + w_y^z \tag{2.11}$$

$$k_1 = 1 + w_z w_y^x \frac{k_1}{k_0} + w_y^z \frac{k_1}{k_0}. \tag{2.12}$$

The following proposition characterizes the equilibrium.

Proposition 3. *There is a unique equilibrium, where:*

$$y = k_0 \theta + k_1 \epsilon_y$$

and k_0 and k_1 solve (2.11) and (2.12), respectively. At the equilibrium solution, the informativeness of the public signal, y_2, about the fundamental, θ, is increasing in the precision of type z's private information, τ_z $\left(\frac{\partial k_0/k_1}{\partial \tau_z} > 0\right)$.

Proof. See Appendix A. □

Proposition 3 shows that the informativeness of the public signal, y_2, increases with τ_z. This, in turn, drives down uncertainty about the underlying fundamental. To understand the intuition behind this result, combine (2.9) with the definition of the public signal y_2 to get:

$$\begin{aligned} y_2 &= \bar{\mathbb{E}}_x\left[\mathbb{E}_z\left(\theta\right)\right] + \epsilon_y \\ &= \bar{\mathbb{E}}\left[w_z z + w_y^z \tilde{y}_2\right] + \epsilon_y \\ &= w_z \bar{\mathbb{E}}\left[\theta\right] + w_y^z \tilde{y}_2 + \epsilon_y. \end{aligned}$$

Increasing the precision of type z individuals' private information increases the weight, w_z, they attach to it in their own signal extraction problem. A larger share of their expectation is therefore driven by variations in their private information, which reflects the unknown fundamental, θ, as opposed to the public signal y_2, observed by everyone. This is the key to understanding the increase in the informativeness of y_2. When inferring type z individuals' beliefs, type x agents will, to a greater extent, have to infer the value of the underlying fundamental rather than the already known public signal. This, in turn, increases, in equilibrium, the informativeness of the public signal, driving down uncertainty about θ.

The outcome of this increase in the precision of private information is similar to the classical learning externality that exists in games where the endogenous signal only reflects first-order beliefs (see, for instance, Grossman (1978) and Amador and Weill (2010)). This externality says that the more precise private information people receive, the more they will impart their private information – the truly new bit of information that other people could learn – into the endogenous public signal, increasing its informativeness. The above shows that for an endogenous public signal comprised of only a higher-order belief a similar effect exist. But instead of imparting directly more private information into the public signal, individuals here expect other people to rely more on their private information as its precision increases. This increases the share of the variation of those people's expectation driven by the fundamental, θ, compared to the public signal itself. And that is what, in turn, induces others (of type x) to forecast θ to a larger extent rather than the known public signal, increasing its informativeness.

That said, despite the decrease in uncertainty about the underlying fundamental, type x agents' uncertainty about type z's beliefs can actually increase. This may at first seem counter-intuitive: x^i knows more about the fundamental and, at the same time, the variance of the error term in type z's private information falls. Both effects, all else equal, contribute to a decline in x^i's uncertainty about type z's beliefs. But increasing the precision of type z's private information also has another effect: it implies that they will update their beliefs by more for any given realization of their private signal – the same feature that drove the decrease in uncertainty about θ. This effect, unlike the two others, works in the opposite direction; it increases, all else equal, other people's uncertainty about type z's expectation. The reason being that type z individuals now update their beliefs relatively more using private information – information that others do not observe – rather than the public information, observed by everyone. In fact, as the following Proposition demonstrates, this third effect dominates when $\tau_\theta = \tau_x = \tau_y$.

Proposition 4. *Agent x^i's uncertainty about the underlying fundamental, θ, equals:*

$$MSE_x^i(\theta) = \frac{1}{\tau_\theta + \tau_x + (k_0/k_1)^2 \tau_y}.$$

Uncertainty about θ is decreasing in the precision of type z's private information, τ_z $\left(\partial MSE_x^i(\theta)/\partial \tau_z < 0\right)$. Uncertainty about type z's expectation of θ can, however, be increasing in τ_z. In fact, for $\tau_\theta = \tau_x = \tau_y$:

$$\left.\frac{\partial MSE_x^i[\mathbb{E}_z(\theta)]}{\partial \tau_z}\right|_{\tau_z=\tau_\theta=\tau_y} > 0,$$

where

$$MSE_x^i\left[\mathbb{E}_z(\theta)\right] = \left(\frac{k_0}{k_1}\right)\left[\frac{1}{\tau_x} + \frac{1}{\tau_\theta + \tau_z + (k_0/k_1)^2\,\tau_y}\right].$$

Proof. See Appendix A. □

Corollary 2. *There exists a value of $\gamma \in \mathbb{R}_+$ such that the utility of type x agents is decreasing in the precision of type z's private information, τ_z.*

Proof. See Appendix A. □

Proposition 4 shows that increasing the precision of people's private information can actually *decrease* how much the endogenous public signal reflects their beliefs. If those people's beliefs matter sufficiently for the determination of, say, asset prices – as the next section indeed shows can be the case – more precise private information can decrease "the informational efficiency of prices" (how good they are at predicting future values) and thus decrease welfare (see also Corollary 2). Moreover, this decrease occurs despite everyone in the economy knowing more about the underlying fundamental, θ.[7]

The simple prediction game used so far is highly abstract. To analyze whether there can indeed be a trade-off between knowing more about a fundamental and knowing more about future asset prices, we need to move beyond this simple model and investigate how the information content of prices change when varying the commonality and precision of information in a standard asset pricing model. The next Section starts with this task. I show how a price encompassing (2.1) and (2.7) is indeed the equilibrium outcome of a standard dispersed information asset pricing model.

3 A Dispersed Information Asset Pricing Model

The asset pricing model I use to illustrate the transmission of information through market-clearing prices resembles Futia (1981), Admati (1985) Singleton (1987), Rondina and Walker (2012), amongst others. I extend the baseline asset pricing model presented in these papers along two directions: I allow for (1) differences in the precision of private information across traders and (2) for a *"non-atomistic"* trader type.

There is a large number of types of traders, indexed by $i \in \{0, 1, 2, \dots N\}$. Each type of trader comprises of a continuum of measure m^i of one-period lived individuals. At each moment in time, the competitive trader $j \in [0, m^i]$ of type $i \in \{0, 1, 2, \dots N\}$ can divide his wealth between two different assets: a risky asset with market-clearing price p_t and coupon payment c_t and a risk-free asset with return r. The wealth of trader (j, i) therefore evolves according to:

$$w_{t+1}^{(j,i)} = x_t^{(j,i)}\left(p_{t+1} + c_{t+1}\right) + \left(w_t^{(j,i)} - x_t^{(j,i)} p_t\right)(1+r), \tag{3.1}$$

where $w_t^{(j,i)}$ denotes the trader's wealth at time t and $x_t^{(j,i)}$ his demand for the risky asset. The competitive trader chooses $x_t^{(j,i)}$ to maximize his expected utility, assuming a CARA functional form. His

[7] The increase in higher-order belief uncertainty does again not stop here: it can also becomes more difficult to forecast "the average expectation", due to the increase in uncertainty about individuals of type's beliefs, and so on – exactly as in Subsection 2.1. This will also be important for later.

portfolio choice problem is thus to maximize:

$$-\mathbb{E}_t^i\left[\exp\left(-\gamma w_{t+1}^{(j,i)}\right)\right]$$

or equivalently:

$$\gamma x_t^{(j,i)}\mathbb{E}_t^i\left[p_{t+1}+c_{t+1}\right]+\gamma\left(w_t^{(j,i)}-x_t^{(j,i)}p_t\right)(1+r)-\frac{\left(\gamma x_t^{(j,i)}\right)^2}{2}\mathbb{V}_t^i\left[p_{t+1}+c_{t+1}\right] \tag{3.2}$$

subject to the budget constraint (3.1). $\mathbb{E}_t^i[\cdot]$ denotes the expectation of agent j of type i at time t conditional on all information, $\mathcal{F}_t^i$, available at time t. The solution to this problem yields agent $(j,\, i)$'s demand for the risky asset:

$$x_t^{(j,i)}=x_t^i=\frac{\mathbb{E}_t^i\left[p_{t+1}+c_{t+1}\right]-(1+r)p_t}{\gamma\mathbb{V}_t^i\left[p_{t+1}+c_{t+1}\right]}\equiv\frac{\mathbb{E}_t^i\left[p_{t+1}+c_{t+1}\right]-(1+r)p_t}{\gamma\delta},$$

as all agents of type j face identical problems.[8] The total demand for the risky asset therefore equals the weighted sum of the speculative demand, x_t^i, from each trader type: $\sum_{i=0}^N x_t^i\phi(i)$, where $\phi(\cdot)$ denotes the density of types.

Like in Singleton (1987), I assume that the supply of the asset depends linearly on the price, p_t, the underlying fundamental, θ_t, and a "noise shock", $\xi_t\sim WN\left(0,\,\sigma_\xi^2\right)$. Hence:

$$s_t=p_t+\theta_t-\xi_t.$$

Market clearing:

$$s_t=\sum_{i=0}^N x_t^i\phi(i)$$

now yields the equilibrium asset price:

$$\begin{aligned}p_t &= \beta\sum_{i=0}^N\mathbb{E}_t^i\left[p_{t+1}+c_{t+1}\right]\phi(i)-\lambda\theta_t+\lambda\xi_t,\\ &\equiv \beta\bar{\mathbb{E}}_t\left[p_{t+1}+c_{t+1}\right]-\lambda\theta_t+\lambda\xi_t,\end{aligned} \tag{3.3}$$

where $\bar{\mathbb{E}}_t\left[p_{t+1}+c_{t+1}\right]\equiv\sum_{i=0}^N\mathbb{E}_t^i\left[p_{t+1}+c_{t+1}\right]\phi(i)$ denotes the average weighted expectation at time t of the asset price and the coupon payment at time $t+1$ and:

$$\begin{aligned}\beta &\equiv \frac{1}{(1+r)+\gamma\delta}\\ \lambda &\equiv \gamma\delta\beta.\end{aligned}$$

[8] Stationarity of the shocks implies that $\delta^i\equiv\mathbb{V}_t^i\left[p_{t+1}+c_{t+1}\right]$ is constant. That said, δ^i is clearly an endogenous object, and as shown by McCafferty and Driskill (1980) several possible stationary equilibria characterized by different values of δ^i could therefore exists. In line with most of the recent literature on the topic (see, for instance, Walker (2007)), I choose for ex-positional reasons to abstract from this complication and set $\delta^i=\delta\in\mathbb{R}_+$ when describing the model and the solution technique. The endogeneity of δ^i and its possible dependence on trader types is handled using the procedure described in Nimark (2011).

The random supply process and the coupon payments, for simplicity, follow stationary AR(1)s. Thus:

$$\theta_t = \rho\theta_{t-1} + \eta_t, \quad \eta_t \sim WN\left(0,\, \sigma_\eta^2\right),\ |\rho| < 1 \tag{3.4}$$

$$c_t = \bar{c} + \psi c_{t-1} + e_t, \quad e_t \sim WN\left(0,\, \sigma_e^2\right),\ |\psi| < 1 \tag{3.5}$$

None of the conclusions that follow depend crucially on the assumptions regarding the stochastic processes for c_t and θ_t – they are merely chosen to simplify the algebra.

3.1 Information and Density Structure

To close the model, I need to specify two further objects: the information structure and the relative size of trader types. The information structure in the model is as follows: All traders observe the coupon payment, c_t, and the endogenous asset price, p_t, which given (3.3) is a noisy public signal of the underlying fundamental, θ_t. In addition, each trader observes an exogenous type-specific signal of the fundamental:

$$s_t^i = \theta_t + \epsilon_t^i, \quad \epsilon_t^i \sim WN\left(0,\, \sigma_{\epsilon^i}^2\right),$$

with signal precision $\tau_s^i \equiv 1/\sigma_{\epsilon^i}^2$, where $\sigma_{\epsilon^i}^2 = \sigma_\epsilon^2$ for $i \neq 0$. In any period t, the information set of trader $(j,\, i)$ is therefore given by:

$$\mathcal{F}_t^i = \left\{p_{t-l},\, s_{t-l}^i,\, c_{t-l}\right\}_{l=0}^\infty,$$

To keep the exposition as tractable as possible, I assume that $m^i = \bar{m}$ for $i \in \{1,\, 2, \ldots N\}$. Combining the assumption of equal mass of trader types $i \in \{1,\, 2, \ldots N\}$ with the Strong Law of Large Numbers simplifies the exposition below considerably by securing that it is only type zero traders' idiosyncratic shock, ϵ_t^0, that affects the market-clearing price: by the Strong Law of Large Numbers $\sum_{i=1}^N \epsilon_t^i \phi(i) = \bar{m}\sum_{i=1}^N \epsilon_t^i = 0$. And hence only type zero traders' signals that are correlated, and hence necessarily have a different degree of signal precision (see Hellwig (1980) and Section 4).[9] The solution technique developed in the following Section to solve the model can, however, straightforwardly be extended to also allow other traders type-specific shock to affect the market-clearing price, and thus have varying degrees of signal precision across the population.

3.2 Dynamic Higher-Order Expectations

Iterating (3.3) forward gives:

$$p_t = \frac{\beta}{1-\beta}\bar{c} + \frac{\beta\psi}{1-\beta\psi}c_t - \lambda\sum_{k=0}^\infty \beta^k \theta_{t+k|\ldots|t}^{(k)} + \lambda\xi_t, \tag{3.6}$$

where, using the same notation as in Nimark (2011), $\theta_{t+k|\ldots|t}^{(k)} \equiv \bar{\mathbb{E}}_t\left[\theta_{t+k|\ldots|t+1}^{(k-1)}\right]$ with $\theta_{t|t}^{(0)} \equiv \theta_t$. In words, $\theta_{t+k|\ldots|t}$ denotes the dynamic (average) higher-order expectation: the average expectation held in period t of the average expectation held in $t+1$, and so on, of the fundamental, θ_{t+k}, in period $t+k$. The

[9] The word "necessarily" here refers to the assumption that $\sigma_{\epsilon^i} = \sigma_\epsilon$ for $i \neq 0$. As I will show in Section 4, even for $\sigma_{\epsilon^0} = \sigma_\epsilon$, the impact of type zero traders' idiosyncratic shock on the asset price implies that their signal precision will differ compared to traders of type $i \neq 0$.

equilibrium price is therefore a weighted average of the dynamic higher-order expectations of θ_t, the coupon payment, c_t, as well as the idiosyncratic supply disturbance. Equation (3.6) presents the asset pricing equation used in the subsequent sections.

3.3 Comparison and Discussion

Like the abstract public signals used in Section 2, the equilibrium asset price in (3.6) is a signal of the underlying fundamental but also, importantly, the average expectation of that fundamental, the average expectation of the average expectation of that fundamental, and so on. The equilibrium asset price combines different trader's beliefs as well as what traders are thinking about each other. This creates the potential for a trade-off between how much the asset price reflects the underlying fundamental and how much it reflects others beliefs of that fundamental.

While the above setup is admittedly highly stylized – the commonality and relative precision of information are controlled directly by just two parameters, $m^0 = m$ and $\sigma^2_{\epsilon^0}$ – it does capture certain basic feature of most financial markets. On the one hand, several financial markets are described by a (global) banking sector that, due to the interconnectedness and the nature of its cross-collateralized lending, faces highly correlated shocks that can matter for the market-clearing price (see Yuan (2005)). On the other hand, these markets also consist of numerous types of both domestic and foreign investment funds, hedge funds, private market participants as well as regional banks, whose type-specific shocks (mostly) can be thought of as canceling out in aggregate.

The above framework also allows explicitly for differences in the degree of signal precision among traders, similar to the setups adopted by Bernhardt et al. (2010) and Walker (2007). This permits us to analyze how differences in the precision of private information disperse through market-clearing prices – and crucially how first- and higher-order beliefs alter this mechanism. The standard assumption in the literature (see e.g. Admati (1985)), that less informed traders behave as white noise processes, can in our framework be thought of as the limiting case, where the less informed traders signal is completely uninformative of the underlying fundamental. As described in the Introduction, differences in the degree of signal precision can, in turn, be rationalized by differences in the degree of Rational Inattention (see Sims (2010) for an overview). But for less informed agents to behave as white noise processes would require them to allocate zero capacity to tracking changes in financial markets – a somewhat contentious assumption.

4 An Infinite State-Space Solution

This Section and the next present the solution to our asset pricing problem. In this Section, I first solve for the full-information case, as it provides a natural benchmark against which we can asses the impact of dispersed information and heterogeneous signal precision. Subsequently, I outline a two-part solution strategy for the dispersed asymmetric information case. The strategy used is in many ways similar to Nimark (2011), but generalizes the solution technique to models with asymmetric information and non-atomistic types. As I will show, the solution strategy implies having to solve a particular fixed-point problem of infinite dimension. The next Section therefore discusses how to find a computationally convenient approximate solution to this fixed-point problem via truncating the state-space.

4.1 The Full Information Case

To solve the model, we need to find an expression for the average expected next period price, $\bar{\mathbb{E}}_t\left[p_{t+1}\right]$, in (3.3). Under full information and rational expectations, that is however easily found, as the expectations of agents always coincide (see Allen et al. (2006)). The Law of Iterated Expectations therefore implies that (3.6) simplifies to:

$$\begin{aligned} p_t^{fi} &= \frac{\beta}{1-\beta}\bar{c} + \frac{\beta\psi}{1-\beta\psi}c_t - \lambda\sum_{k=0}^{\infty}\beta^k\mathbb{E}_t\left[\theta_{t+k}\right] + \lambda\xi_t \\ &= \frac{\beta}{1-\beta}\bar{c} + \frac{\beta\psi}{1-\beta\psi}c_t - \frac{\lambda}{1-\beta\rho}\theta_t + \lambda\xi_t, \end{aligned} \tag{4.1}$$

which gives us our full-information solution.

4.2 A Two-part Procedure with an Infinite State Vector

To solve the model, given by equation (3.6), under dispersed asymmetric information, I conjecture (and will later verify) that the solution takes the following form:

$$p_t = \frac{\beta}{1-\beta}\bar{c} + \frac{\beta\psi}{1-\beta\psi}c_t + \alpha' X_t^{(0:\infty)} + \lambda\xi_t \tag{4.2}$$

$$X_t^{(0:\infty)} = MX_{t-1}^{(0:\infty)} + N\omega_t, \quad \omega_t = \left[\eta_t,\, \epsilon_t^0,\, \xi_t\right]', \tag{4.3}$$

where $X_t^{(0:\infty)}$ denotes the relevant state vector. In words, I conjecture that the price can be described by a linear combination of the components in $X_t^{(0:\infty)}$, and that the state follows a VAR(1). To solve for the state transition equation, we first need to find the elements included in $X_t^{(0:\infty)}$. Proposition 5 provides the composition of the infinite state vector:

Proposition 5. *The state vector, $X_t^{(0:\infty)}$, for the asset pricing model is given by:*

$$X_t^{(0:\infty)} = \begin{bmatrix} X_t^{(0)} \\ X_t^{(1)} \\ \vdots \end{bmatrix}, \quad X_t^{(k)} = \begin{bmatrix} \bar{\mathbb{E}}_t X_t^{(k-1)} \\ \mathbb{E}_t^0 A_k X_t^{(k-1)} \end{bmatrix}, \quad k \geq 1 \tag{4.4}$$

where A_k denotes an annihilation matrix that removes the last f_k rows from $\mathbb{E}_t^0\left[X_t^{(k-1)}\right]$ and $X_t^{(0)} = \theta_t$.[10]

Proof. See Appendix A under the heading "*The State Conjecture and The Infinite Regress*". □

In Proposition 5, A_k removes the last f_k rows from $\mathbb{E}_t^0\left[X_t^{(k-1)}\right]$ as these are superfluous: they consist of elements given by type zero agents' expectation of their own expectation, which, by the Law

[10] To get a better feel for the structure of $X_t^{(0:\infty)}$, I simplify the notation by following Woodford (2002) and denote current period average higher-order expectations as: $\theta_t^{(k)} \equiv \bar{\mathbb{E}}_t\left[\theta_t^{(k-1)}\right]$. Similarly, I denote the systemically important types expectation of, for instance, θ_t as: $\theta_t^0 \equiv \mathbb{E}_t^{(0)}\left[\theta_t\right]$. "Cross-expectations", such as type zero traders' expectation of the

of Iterated Expectations, can be reduced to their own expectation, already defined higher up in the hierarchy. Interestingly, f_k follows the Fibonacci Sequence and therefore evolves according to Binet's formula: $f_k = \frac{\phi^k - \psi^k}{\phi - \psi}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Given (4.4), an alternative, and computationally more useful, expression for the hierarchy of expectations is:

$$X_t^{(0:\infty)} = H \begin{bmatrix} \theta_t \\ \bar{\mathbb{E}}_t X_t^{(0:\infty)} \\ \mathbb{E}_t^0 A X_t^{(0:\infty)} \end{bmatrix}, \tag{4.5}$$

where $H : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is a permutation matrix that re-orders the elements of the right-hand side vector, while A is block-diagonal matrix that has the matrices A_k, $k = 0,\ 1,\ ...$ along its diagonal.[11]

Given the conjecture in (4.2), (4.3) and the definition of the state in Proposition 5, I now outline a two-part solution strategy that can be used to characterize the solution to the asset pricing problem with an infinite state vector.

4.2.1 Part-1-of-2: The Coefficient Vector

The first part in the procedure consists of finding the price of the asset given a law of motion for the state. To find the vector of coefficients, α', I use the Method of Undetermined Coefficients. Using the conjecture in (4.2) and (4.3), we get from (3.3) that:

$$p_t = \frac{\beta}{1-\beta}\bar{c} + \frac{\beta\psi}{1-\beta\psi}c_t + \beta\alpha' M \bar{\mathbb{E}}_t\left[X_t^{(0:\infty)}\right] - \lambda e_1' X_t^{(0:\infty)} + \lambda\xi_t.$$

Using (4.5), $\bar{\mathbb{E}}_t\left[X_t^{(0:\infty)}\right]$ can be expressed as (where I suppress the size of the matrices for ease of notation):

$$\bar{\mathbb{E}}_t\left[X_t^{(0:\infty)}\right] = \begin{bmatrix} \underline{0} & I & \underline{0} \end{bmatrix} H^{-1} X_t^{(0:\infty)}.$$

Thus implying that the market-clearing price can be written as:

average expectation, are accordingly defined as: $\theta_t^{(0)(1)} \equiv \mathbb{E}_t^{(0)}\bar{\mathbb{E}}_t\left[x_t\right]$. Using this notation, allows us to express $X_t^{(0:\infty)}$ as:

$$X_t^{(0:\infty)} = \begin{bmatrix} \theta_t \\ \hline \theta_t^{(1)} \\ \theta_t^{(0)} \\ \hline \theta_t^{(2)} \\ \theta_t^{(1)(0)} \\ \theta_t^{(0)(1)} \\ \hline \vdots \end{bmatrix},$$

where the horizontal bars separate the orders of expectation, k. This representation of $X_t^{(0:\infty)}$ perhaps more easily shows how the number of elements in each order of expectation increase according to the Fibonacci Sequence.

[11] To see the role of the matrix A, first consider the state vector: $X_t^{(0:\infty)} = \begin{bmatrix} \theta_t \,| & \theta_t^{(1)} & \theta^0 & |\, \theta^{(2)} & \theta^{(1)0} & \theta^{0(1)} \,| & \dots \end{bmatrix}'$, where the vertical bar separates the orders of expectation. $\mathbb{E}_t^0\left[X_t^{(0:\infty)}\right]$ equals: $\mathbb{E}_t^0\left[X_t^{(0:\infty)}\right] = \begin{bmatrix} \theta_t^0 \,| & \theta_t^{0(1)} & \theta^0 & |\, \theta^{0(2)} & \theta^{0(1)0} & \theta^{0(1)} \,| & \dots \end{bmatrix}'$. Note how $\mathbb{E}_t^0\left[X_t^{(0:\infty)}\right]$ contains, for instance, the elements θ_t^0 and $\theta_t^{0(1)}$ twice. The role of the matrix A is to zero out the elements of $X_t^{(0:\infty)}$ that generate these redundant expectations. $\mathbb{E}_t^0\left[AX_t^{(0:\infty)}\right]$ thus equals: $\mathbb{E}_t^0\left[AX_t^{(0:\infty)}\right] = \begin{bmatrix} \theta_t^0 \,| & \theta_t^{0(1)} \,| & \theta^{0(2)} & \theta^{0(1)0} & | & \dots \end{bmatrix}'$.

$$p_t = \frac{\beta}{1-\beta}\bar{c} + \frac{\beta\psi}{1-\beta\psi}c_t + \left[\alpha'\beta M \begin{bmatrix} \underline{0} & I & \underline{0} \end{bmatrix} H^{-1} - \lambda e_1'\right] X_t^{(0:\infty)} + \lambda\xi_t.$$

Matching coefficients gives that α' equals:

$$\alpha' = -\lambda e_1' \left[I - \beta M \begin{pmatrix} \underline{0} & I & \underline{0} \end{pmatrix} H^{-1}\right]^{-1}. \tag{4.6}$$

4.2.2 Part-2-of-2: The Law of Motion for the Infinite State

From (4.5) we see that the state consists of three components: (i) the actual supply disturbance; (ii) the average expectation of the hierarchy; and lastly (iii) traders of type zero's expectation of (some elements of) the same hierarchy. The Kalman Filter therefore plays it usual dual role: it both determines how the individual traders estimate the state as well as the law of motion of the very same state that the traders are estimating. To find the coefficient matrices in the law of motion for the state, M and N, the strategy is to first derive $\bar{\mathbb{E}}_t\left[X_t^{(0:\infty)}\right]$ and $\mathbb{E}_t^0\left[AX_t^{(0:\infty)}\right]$ as linear functions of $X_{t-1}^{(0:\infty)}$ and ω_t. Subsequently, these expressions can be stacked via (4.5) to give a VAR(1) form for the evolution of the state – exactly the same form as our conjecture in (4.3). Last, equating coefficients will give the solution for M and N. As inputs, we will also need α', derived in the previous part.

Beginning with the first step, deleting the first row and column of H, and denoting the new matrix as $\tilde{H}$, gives us that:

$$X_t^{(1:\infty)} = \tilde{H} \begin{bmatrix} \bar{\mathbb{E}}_t X_t^{(0:\infty)} \\ \mathbb{E}_t^0 AX_t^{(0:\infty)} \end{bmatrix}. \tag{4.7}$$

The link between the expectation of type zero traders' estimate of the aggregate state, $\mathbb{E}_t^0\left[X_t^{(0:\infty)}\right]$, and $\mathbb{E}_t^0\left[AX_t^{(0:\infty)}\right]$, which features in (4.7), can be expressed using yet another re-ordering matrix, $R : \mathbb{R}^\infty \to \mathbb{R}^\infty$, defined by:

$$\mathbb{E}_t^0\left[X_t^{(0:\infty)}\right] \equiv R\mathbb{E}_t^0\left[AX_t^{(0:\infty)}\right]. \tag{4.8}$$

While A annihilates the rows of the state that contain as the first element type zero traders' expectation, R re-orders them back. That way, $\mathbb{E}_t^0\left[AX_t^{(0:\infty)}\right]$ does not contain any superfluous element. $\mathbb{E}_t^0\left[X_t^{(0:\infty)}\right]$, on the other hand, will contain, for instance, type zero traders' expectation of the actual state twice. Also, due to the re-ordering of the elements: $RA \neq I$.

To derive expressions for $\bar{\mathbb{E}}_t\left[X_t\right]$ and $\mathbb{E}_t^0\left[AX_t^{(0:\infty)}\right]$ as linear functions of $X_{t-1}^{(0:\infty)}$ and ω_t, we need to solve traders' signal extraction problem. In terms of the state, type i's observation equation takes the form:

$$\begin{aligned} z_t^i &= \begin{bmatrix} s_t^i \\ p_t \end{bmatrix} = \begin{bmatrix} e_1' \\ \alpha' \end{bmatrix} X_t^{(0:\infty)} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \epsilon_t^i + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \eta \\ \epsilon_t^0 \\ \xi_t \end{bmatrix}, \\ &\equiv LX_t^{(0:\infty)} + Q_\epsilon \epsilon_t^i + Q_\omega \omega_t, \quad i \in \{0, 1, 2, \ldots N\} \end{aligned} \tag{4.9}$$

where e_j denotes the column vector of $dim\left(X_t^{(0:\infty)}\right)$ where the jth entry is one and all other entries are zero.

To form minimum mean-squared-error estimates of the current state, traders use the Kalman Filter to update their forecasts. Signal extraction is therefore done via the updating equation:

$$\mathbb{E}_t^i\left[X_t^{(0:\infty)}\right] = M\mathbb{E}_{t-1}^i\left[X_{t-1}^{(0:\infty)}\right] + K^i\left(z_t^i - \mathbb{E}_{t-1}^i\left[z_t^i\right]\right), \tag{4.10}$$

where K^i denotes the vector of Kalman Gains given by the linear projection formula:

$$K^i \equiv \mathbb{E}\left[\left(X_t^{(0:\infty)} - \mathbb{E}_{t-1}^i\left[X_t^{(0:\infty)}\right]\right)\left(z_t^i - \mathbb{E}_{t-1}^i\left[z_t^i\right]\right)'\right]\mathbb{E}\left[\left(z_t^i - \mathbb{E}_{t-1}^i\left[z_t^i\right]\right)\left(z_t^i - \mathbb{E}_{t-1}^i\left[z_t^i\right]\right)'\right]^{-1}.$$

Even if the accuracy of the type-specific signals were the same across types $\left(\sigma_{\epsilon^0}^2 = \sigma_\epsilon^2\right)$, since the systemically important agents' idiosyncratic shock in equilibrium affects the asset price (see below), traders of type zero will on average receive a signal vector with a different degree of informativeness about the underlying fundamental [see (4.9)]. Thus, K^i will differ between type $i = 0$ and the remaining $N-1$ types, regardless of whether $\sigma_{\epsilon^0}^2 < \sigma_\epsilon^2$ or $\sigma_{\epsilon^0}^2 = \sigma_\epsilon^2$. Specifically, for type $i \in \{1, 2, \dots N\}$, the Kalman Gain will be given by:

$$\begin{aligned} K^i &\equiv K = \left(PL' + N\Sigma Q_\omega'\right) \\ &\cdot \left(LPL' + \sigma_\epsilon^2 Q_\epsilon Q_\epsilon' + Q_\omega\Sigma Q_\omega' + LN\Sigma Q_\omega' + Q_\omega\Sigma N'L'\right)^{-1} \end{aligned} \tag{4.11}$$

$$P^i \equiv P = M\left[P - K\left(PL' + N\Sigma Q_\omega'\right)'\right]M' + N\Sigma N', \tag{4.12}$$

where $\Sigma \equiv diag\left(\sigma_\eta^2, \sigma_{\epsilon^0}^2, \sigma_\xi^2\right)$. While for $i = 0$:

$$\begin{aligned} K^0 &= \left(P^0L' + N\Sigma Q_0'\right) \\ &\cdot \left(LP^0L' + Q_0\Sigma Q_0' + LN\Sigma Q_0' + Q_0\Sigma N'L'\right)^{-1} \end{aligned} \tag{4.13}$$

$$P^0 = M\left[P^0 - K^0\left(P^0L' + N\Sigma Q_0'\right)'\right]M' + N\Sigma N', \tag{4.14}$$

where

$$Q_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

The distinction between K and K^0 expresses the difference in average signal precision between the two sets of agents, caused by the effect of type zero traders' idiosyncratic shock on the market-clearing price, as well as the potential asymmetry in the accuracy of the type-specific signals.

Whether with identical signal precision in their type-specific signals traders of type $i = 0$ are better or worse informed about the next period state, X_{t+1}, than traders of type $j \neq 0$ depends on two offsetting effects: All else equal, the greater correlation between the elements of their signal vector – caused by type $i = 0$ traders type-specific shock affecting the market-clearing price – causes the systemically important agents to be relatively more confused about whether a change in their signal vector is due to a change in θ_t or a change in ϵ_t^0. The increased correlation therefore implies a greater mean-squared error of next period state, P^0. But, as some of the noise in the price is caused by the error in type zero traders' type-specific signal, they will also be better informed about the indirect effect of ϵ_t^0 on p_t, implying, all else equal, a smaller mean-squared error. Which of these two effects dominates determines whether, for $\sigma^2_{\epsilon^0} = \sigma^2_\epsilon$, traders of type $i = 0$ or type $j \neq 0$ are better informed.

Taking the weighted average expectation across (4.10), using the density function $\phi(\cdot)$, gives us that:

$$\begin{aligned} \bar{\mathbb{E}}_t \left[X_t^{(0:\infty)} \right] &= M \bar{\mathbb{E}}_{t-1} \left[X_{t-1}^{(0:\infty)} \right] + K \left(\bar{z}_t - LM \bar{\mathbb{E}}_{t-1} \left[X_{t-1}^{(0:\infty)} \right] \right) \\ &+ m \left(K^0 - K \right) \left(z_t^0 - LM \mathbb{E}_{t-1}^0 \left[X_{t-1}^{(0:\infty)} \right] \right). \end{aligned} \tag{4.15}$$

While type zeros' expectation equals:

$$\mathbb{E}_t^0 \left[X_t^{(0:\infty)} \right] = M \mathbb{E}_{t-1}^0 \left[X_{t-1}^{(0:\infty)} \right] + K^0 \left(z_t^0 - LM \mathbb{E}_{t-1}^0 \left[X_{t-1}^{(0:\infty)} \right] \right). \tag{4.16}$$

Using (4.7) and (4.8), we can rewrite the average expectation of the state as:

$$\begin{aligned} \bar{\mathbb{E}}_t \left[X_t^{(0:\infty)} \right] &= [I - KL] M \begin{bmatrix} I & \underline{0} \end{bmatrix} \tilde{H}^{-1} X_{t-1}^{(1:\infty)} \\ &+ K \bar{z}_t + m \left(K^0 - K \right) \left(z_t^0 - LMR \begin{bmatrix} \underline{0} & I \end{bmatrix} \tilde{H}^{-1} X_{t-1}^{(1:\infty)} \right). \end{aligned}$$

From the definition of the private signal, (4.9), the price conjecture, (4.2), and that $\sum_{i=1}^N \epsilon_t^i = 0$ by the Strong Law of Large Numbers, the average signal, $\bar{z}_t$, can be written as:

$$\bar{z}_t = LX_t^{(0:\infty)} + \bar{Q}\omega_t, \quad \bar{Q} = \begin{bmatrix} 0 & m & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Substituting the average signal into our expression for the average expectation of the state, along with traders of type zero's signal, z_t^0, gives:

$$\begin{aligned} \bar{\mathbb{E}}_t \left[X_t^{(0:\infty)} \right] &= \left\{ [I - KL] M \begin{bmatrix} I & \underline{0} \end{bmatrix} + m \left(K^0 - K \right) LMR \begin{bmatrix} \underline{0} & I \end{bmatrix} \right\} \tilde{H}^{-1} X_{t-1}^{(1:\infty)} \\ &+ \left[K + m \left(K^0 - K \right) \right] LMX_{t-1}^{(0:\infty)} \\ &+ \left[K(LN + \bar{Q}) + m \left(K^0 - K \right) (LN + Q_0) \right] \omega_t, \end{aligned}$$

or

$$\bar{\mathbb{E}}_t \left[X_t^{(0:\infty)} \right] \equiv \bar{M}_1 X_{t-1}^{(1:\infty)} + \bar{M}_2 X_{t-1}^{(0:\infty)} + \bar{N}\omega_t, \tag{4.17}$$

where the matrices $\bar{M}_1$, $\bar{M}_2$ and $\bar{N}$ are implicitly defined.

Similarly, inserting type zero's signal, in addition to using (4.7) and (4.8), allows us to write (4.16) as:

$$\begin{aligned} \mathbb{E}_t^0 \left[AX_t^{(0:\infty)} \right] &= R_{left}^{-1} \left[I - K^0 L \right] MR \begin{bmatrix} \underline{0} & I \end{bmatrix} \tilde{H}^{-1} X_{t-1}^{(1:\infty)} \\ &+ R_{left}^{-1} K^0 LMX_{t-1}^{(0:\infty)} + R_{left}^{-1} K^0 \left(LN + Q_0 \right) \omega_t, \end{aligned}$$

where R_{left}^{-1} denotes the left inverse of R. Alternatively, this can be written as:

$$\mathbb{E}_t^0\left[AX_t^{(0:\infty)}\right] \equiv M_1^0 X_{t-1}^{(1:\infty)} + M_2^0 X_{t-1}^{(0:\infty)} + N^0\omega_t, \tag{4.18}$$

where the matrices M_1^0, M_2^0 and N^0 are again implicitly defined.

The final steps to get the conjectured form (4.3) are to collect terms, stack via (4.7) and append the actual supply disturbance process (3.4). Stacking using (4.7) gives:

$$\begin{aligned} X_t^{(1:\infty)} &= \tilde{H}\begin{bmatrix} \bar{\mathbb{E}}_t X_t^{(0:\infty)} \\ \mathbb{E}_t^0 A X_t^{(0:\infty)} \end{bmatrix} \\ &= \tilde{H}\begin{bmatrix} \bar{M}_1 \\ M_1^0 \end{bmatrix} X_{t-1}^{(1:\infty)} + \tilde{H}\begin{bmatrix} \bar{M}_2 \\ M_2^0 \end{bmatrix} X_{t-1}^{(0:\infty)} + \tilde{H}\begin{bmatrix} \bar{N} \\ N^0 \end{bmatrix}\omega_t. \end{aligned}$$

Finally, appending the supply disturbance process verifies the conjectured VAR(1) form:

$$\begin{aligned} X_t^{(0:\infty)} &= \begin{bmatrix} \theta_t \\ X_t^{(1:\infty)} \end{bmatrix} \\ &= \begin{bmatrix} \rho & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}\begin{bmatrix} \theta_{t-1} \\ X_{t-1}^{(1:\infty)} \end{bmatrix} + \begin{bmatrix} 0 & \underline{0} \\ \underline{0} & \tilde{H}\begin{pmatrix} \bar{M}_1 \\ M_1^0 \end{pmatrix} \end{bmatrix}\begin{bmatrix} \theta_{t-1} \\ X_{t-1}^{(1:\infty)} \end{bmatrix}. \\ &+ \begin{bmatrix} \underline{0} \\ \tilde{H}\begin{pmatrix} \bar{M}_2 \\ M_2^0 \end{pmatrix} \end{bmatrix}\begin{bmatrix} \theta_{t-1} \\ X_{t-1}^{(1:\infty)} \end{bmatrix} + \begin{bmatrix} e_1' \\ \tilde{H}\begin{pmatrix} \bar{N} \\ N^0 \end{pmatrix} \end{bmatrix}\omega_t \end{aligned}$$

Equating coefficients with our conjecture in (4.3) gives the solution to the coefficient matrices in the law of motion for the state:

$$M = \begin{bmatrix} \rho & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} + \begin{bmatrix} 0 & \underline{0} \\ \underline{0} & \tilde{H}\begin{pmatrix} \bar{M}_1 \\ M_1^0 \end{pmatrix} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \tilde{H}\begin{pmatrix} \bar{M}_2 \\ M_2^0 \end{pmatrix} \end{bmatrix} \tag{4.19}$$

$$N = \begin{bmatrix} e_1' \\ \tilde{H}\begin{pmatrix} \bar{N} \\ N^0 \end{pmatrix} \end{bmatrix}. \tag{4.20}$$

Equation (4.6), (4.19) and (4.20) present an infinite-dimensional mapping from $\{M, N\} \to \{M, N\}$. The fixed-point of this mapping provides the solution to the asset pricing equation by giving us the coefficient matrices for the evolution of the infinite dimensional state, $X_t^{(0:\infty)}$, as well as, via (4.6), the coefficients on the contemporaneous expectation hierarchy in the market-clearing price. The next section discusses how to find an approximate solution to this infinite-dimensional fixed-point problem.

5 A Truncated Approximate Solution

The solution to the model given by (4.2), (4.3), and the fixed-point of (4.19) and (4.20) is unfortunately not practically feasible due to the infinite length of the state vector. In this section, I therefore propose an iterative algorithm providing an approximate solution that truncates the state-space after $\bar{k}$ segments of higher-order expectations. As I will show, increasing $\bar{k}$ allows us to find an arbitrarily precise, and comparatively simple, solution to the asset pricing problem.

5.1 A Truncated Solution:

The rest of this paper focuses on finding a finite dimensional approximation of the equilibrium price under dispersed asymmetric information (3.6) of the form:

$$p_t^{\bar{k}} = \tfrac{\beta}{1-\beta}\bar{c} + \tfrac{\beta\psi}{1-\beta\psi}c_t + \alpha'_{(0:\bar{k})}X_t^{(0:\bar{k})} + \lambda m\epsilon_t^0 + \lambda\xi_t \tag{5.1}$$

$$X_t^{(0:\bar{k})} = M^{\bar{k}}X_{t-1}^{(0:\bar{k})} + N^{\bar{k}}\omega_t, \tag{5.2}$$

where $X_t^{(0:\bar{k})}$ is a vector that includes $\bar{k}$ segments of current period higher-order expectations of θ_t and $\alpha_{(0:\bar{k})}$ a vector that includes the corresponding elements of α. The approach taken here therefore, as in Nimark (2011), approximates the true equilibrium price, p_t, by truncating the state-space.

For the truncated state-space solution, $p_t^{\bar{k}}$, to approximate the market-clearing price, p_t, we require that the variance of the approximation error:

$$\mathbb{V}\left[p_t - p_t^{\bar{k}}\right] \equiv \left[\alpha' - \begin{pmatrix} \alpha'_{(0:\bar{k})} & 0 \end{pmatrix}\right]\mathbb{V}\left[X_t\right]\left[\alpha' - \begin{pmatrix} \alpha'_{(0:\bar{k})} & 0 \end{pmatrix}\right]', \tag{5.3}$$

is asymptotically decreasing towards zero in $\bar{k}$. We therefore require that the added contribution to the dynamics of $p_t^{\bar{k}}$ from including one more order of expectation at some moment declines to zero. The following Propositions establish that this indeed will be the case.

Proposition 6. *For $|\beta\rho| < 1$, there exists, for each $\varepsilon > 0$, a $\bar{n} \in \mathbb{Z}_+$ such that*

$$\max\left|\alpha_{(n-1:n)}\right| < \varepsilon$$

for all $n \geq \bar{n}$.

Proof. See Appendix B. □

Proposition 6 shows that as we increase $\bar{k} \to \infty$, the largest absolute element in segment $\bar{k}$ in α will eventually converge to zero. However, Proposition 6 is not on its own enough to guarantee that the variance of the approximation error declines towards zero in $\bar{k}$. In principle, it could be possible that the variance and covariance elements in $\mathbb{V}\left[X_t\right]$ would increase sufficiently fast in $\bar{k}$ to offset the decrease in the maximum absolute value in $\alpha_{(\bar{k}-1:\bar{k})}$, implying that the added contribution to the dynamic of $p_t^{\bar{k}}$ of an extra order would always increase [see equation (5.3)]. Proposition 7 below reassuringly shows that this kind of explosive behavior cannot be the case.

Proposition 7. *The kth order of the hierarchy is given by:*

$$X_t^{(k)} = \begin{bmatrix} \bar{\mathbb{E}}_t X_t^{(k-1)} \\ \mathbb{E}_t^0 A_k X_t^{(k-1)} \end{bmatrix}, \quad k \geq 1.$$

If $|\rho| < 1$*, then:*

$$\begin{aligned} diag\left\{\mathbb{V}\left[\bar{\mathbb{E}}_t X_t^{(k-1)}\right]\right\} &\leq diag\left\{\mathbb{V}\left[X_t^{(k-1)}\right]\right\} \\ diag\left\{\mathbb{V}\left[\mathbb{E}_t^0 A_k X_t^{(k-1)}\right]\right\} &\leq diag\left\{\mathbb{V}\left[A_k X_t^{(k-1)}\right]\right\}, \end{aligned}$$

where $diag\{\cdot\}$ *denotes the diagonal elements of a matrix.*

Proof. See Appendix B, including Lemma 1. □

Proposition 7 demonstrates that the variance of each component of the hierarchy of expectations is a 'partwise' non-increasing sequence (I will later demonstrate that this sequence also converges).[12] However, unlike Nimark (2011), we cannot claim that each element in the hierarchy is non-increasing: type zero traders' expectation of future values of θ_t might, for instance, be more accurate than the average expectation across *all* types. But what we can conclude is that the variance of 'partwise' elements across orders of the hierarchy will be non-increasing. The intuition behind Proposition 7 follows from the well-known result that the variance of an optimal forecast is no-greater than the variance of the variable being forecasted (see e.g. Hamilton (1994)). As (4.5) shows, each element in $X_t^{(k)}$ for $k \geq 1$ is an expectation of a previous element in the hierarchy. Repeatedly applying the optimal forecast result, using Lemma 1 in the Appendix, establishes Proposition 7.

The next Proposition provides a key result.

Proposition 8. *If* $\beta < 1$ *and* θ_t *is stationary, the variance of the approximation error,* $\mathbb{V}\left[p_t - p_t^{\bar{k}}\right]$*, where* $p_t^{\bar{k}} \equiv \frac{\beta}{1-\beta}\bar{c} + \frac{\beta\psi}{1-\beta\psi}c_t + \alpha'_{(0:\bar{k})}X_t^{(0:\bar{k})} + \lambda\xi_t$*, tends to zero as* $\bar{k} \to \infty$*.*

Proof. See Appendix B. □

Combining Proposition 6 with Proposition 7 we see that if agents discount the future, $\beta < 1$, and the underlying fundamental is stationary, $|\rho| < 1$, then a truncated solution of the form (5.1) and (5.2) will to an arbitrarily accurate degree approximate the dynamics of the market-clearing price, p_t, given by (4.2) and (4.3). Intuitively, Proposition 6 shows that the coefficients in α become smaller and smaller as we increase the order of expectation, k; while Proposition 7 shows that the variance of higher-order expectations are 'partwise' non-increasing. Combining these two results makes it clear that the contribution of each added order of expectation to the dynamics of the asset price must eventually

[12]'Partwise' here refers to the two components of the kth segment of the hierarchy in (4.5): (i) the average expectation of all the elements in the $(k-1)$th order, and (ii) type zero agents' expectation of the elements in the $(k-1)$th order that are average expectations.

decrease to zero. Thus, a truncated state-space solution, $p_t^{\bar{k}}$ and $X_t^{(0:\bar{k})}$, can approximate the true state-space, p_t and X_t.

The crucial assumption that θ_t is stationary is the same assumption used in Kasa et al. (2012), Bernhardt et al. (2010) and Rondina and Walker (2012), who all apply frequency domain methods to find analytical solutions to special cases of the framework presented here, where the number of shocks equals the number of observables. There thus appears to be a tight relationship between the existence of frequency domain solutions to dispersed information models and the applicability of truncation based approaches.[13]

5.2 The Solution Algorithm

The previous subsection discussed the feasibility of an approximate truncated state-space solution of the form (5.1) and (5.2). But it did not discuss how to find a fixed-point of the corresponding finite dimensional version of (4.6), (4.19) and (4.20). This subsection therefore presents an iterative algorithm for the fixed-point problem that truncates the state-space after $\bar{k}$ higher-order expectations. Proposition 9 below establishes that a fixed-point of the mapping $\left\{M_j^{\bar{k}}, N_j^{\bar{k}}\right\} \to \left\{M_{j+1}^{\bar{k}}, N_{j+1}^{\bar{k}}\right\}$ will exist for each $\bar{k} \in \mathbb{Z}_+$.

Proposition 9. *The mapping $\left\{M_j^{\bar{k}}, N_j^{\bar{k}}\right\} \to \left\{M_{j+1}^{\bar{k}}, N_{j+1}^{\bar{k}}\right\}$ is continuous and from a convex compact set onto itself. Brouwer's fixed-point Theorem therefore applies, guaranteeing the existence of a fixed-point.*

Proof. See Appendix B (similar to Nimark (2011)). □

There are two steps in each iteration, j, of the solution algorithm: (i) For a given law of motion for the state, $M_j^{\bar{k}}$ and $N_j^{\bar{k}}$, find the row vector, $\alpha'_{j,(0:\bar{k})}$, that maps the state into the market-clearing price using:

$$\alpha'_{j,(0:\bar{k})} = -\lambda e_1' \left[I_{\bar{k}} - \beta M_j^{\bar{k}} \begin{pmatrix} \underline{0}_{\bar{k}\times 1} & I_{\bar{k}} & \underline{0}_{\bar{k}} \end{pmatrix} H^{-1} \right]^{-1}. \tag{5.4}$$

And (ii), for given $\alpha'_{j,(0:\bar{k})}$, $M_j^{\bar{k}}$ and $N_j^{\bar{k}}$, find the new law of motion for the hierarchy of expectations using the above solution to the coefficient matrices in the state evolution equation:

$$M_{j+1}^{\bar{k}} = \begin{bmatrix} \rho & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} + \begin{bmatrix} 0 & \underline{0} \\ \underline{0} & \tilde{H} \begin{pmatrix} \bar{M}_{j,1}^{\bar{k}} \\ M_{j,1}^{0,\bar{k}} \end{pmatrix} \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \tilde{H} \begin{pmatrix} \bar{M}_{j,2}^{\bar{k}} \\ M_{j,2}^{0,\bar{k}} \end{pmatrix} \end{bmatrix} \tag{5.5}$$

$$N_{j+1}^{\bar{k}} = \begin{bmatrix} \sigma_\eta e_1' \\ \tilde{H} \begin{pmatrix} \bar{N}_j^{\bar{k}} \\ N_j^{0,\bar{k}} \end{pmatrix} \end{bmatrix}, \tag{5.6}$$

where the last rows and/or columns of the matrices have been cropped to make the matrices conformable. In other words, implementing the approximation that expectations of order $k > \bar{k}$ are redundant, and

[13] Appendix B shows that both are related to the existence of $\mathbb{V}[p_t]$.

therefore setting $X_t^{(k)} = \underline{0}_{f_k \times 1}\ \forall k > \bar{k}$. To initialize the algorithm, I use the solution to the full information case, given by (4.1). Appendix E discusses the accuracy of the approximate solution as well as the approach used to determine the maximum order of expectation, $\bar{k}$.

6 The Informativeness of Asset Prices

The welfare of traders in the model depends directly on their ability to correctly anticipate movements in the endogenous asset price (see 3.2). This price is, in turn, a function of two persistent components: the underlying fundamental and (higher-order) expectations of that fundamental. Welfare in the economy therefore depends on traders' ability to accurately infer both of these components of future prices. In this section, I demonstrate that two propositions typically thought to decrease "the informational efficiency of prices" can in fact be welfare enhancing: Both (1) increases in the commonality of information – the proportion of individuals who receive the same private signal – and (2) decreases in the precision of private information improve the ability of traders to accurately forecast the beliefs of others. Since the endogenous asset price to a large extent depends on higher-order expectations, uncertainty about future prices can fall – despite everyone knowing less about movements in the underlying fundamental. Higher-order beliefs play a crucial role in understanding the dissemination of information through equilibrium prices.

To demonstrate these findings, I use the following baseline parameter configuration for the dispersed information asset pricing model:

$$\left\{\bar{k},\, \delta,\, r,\, \gamma,\, \psi,\, \rho,\, \sigma_\eta,\, \sigma_c,\, \sigma_\xi,\, \sigma_\epsilon,\, \sigma_{\epsilon^0},\, m\right\} =$$
$$\{11,\, 0.1,\, 0.01,\, 1,\, 0.5,\, 0.9,\, 0.5,\, 0.5,\, 2,\, 2,\, 2,\, 0.50\}\,.$$

This parametrization corresponds to a situation where the only asymmetry in average signal precision between the different types of agents stems from the effect of type zero traders on the market-clearing price. The mass of type zero traders is as default set to half of the total population. The parameters determining the precision of traders private information are naturally difficult to precisely pin down. The approach taken in the below is to set these parameters such that the signal-to-noise ratio of the private signals equals $\frac{1}{4}$. This "intermediate level" of signal precision increases the role of higher-order beliefs in the determination of asset prices, and hence helps to clearly illustrate the key insights of this paper.[14] The remaining parameters correspond to those used in Nimark (2011). Appendix C investigates the role of higher-order expectations and asymmetric private information for the dynamics of the asset price.

[14]For very precise signals, higher-order expectations are accurately captured by first-order beliefs, thereby not adding much to the dynamics of the price. Similarly, for very imprecise signals, higher-order expectations do not respond significantly to changes in signal values, implying that higher-order thinking does not contribute meaningfully to movements in the endogenous variable. Only for intermediate values of signal precision do higher-order expectations matter greatly for price dynamics (see also Lorenzoni (2009))

6.1 The Commonality of Information

Increasing the commonality of information – captured by the parameter m, denoting the relative size of type zero traders – causes the endogenous asset price to reflect type zero traders' expectation to a larger extent. Proposition 2 and Corollary 1 demonstrated in a simple prediction game that this increase implies a trade-off: On the one hand, because of the decrease in the "wisdom of the crowds" – the extent to which the price combines different traders' private information – the public signal becomes a poorer indicator of the underlying fundamental. This, at the same time, makes it more difficult to infer type $j \neq 0$ traders' beliefs of that fundamental, as they attach relatively more weight to their private information when forming their expectations. However, on the other hand, the price now also contains more precise information about type zero traders' expectation of the fundamental, and thus also of the average expectation of type zero traders' expectation of the fundamental and so on. If these type zero higher-order beliefs matter sufficiently for the endogenous price that people are attempting to predict, welfare can increase – despite everyone knowing less about the shocks affecting the economy.[15]

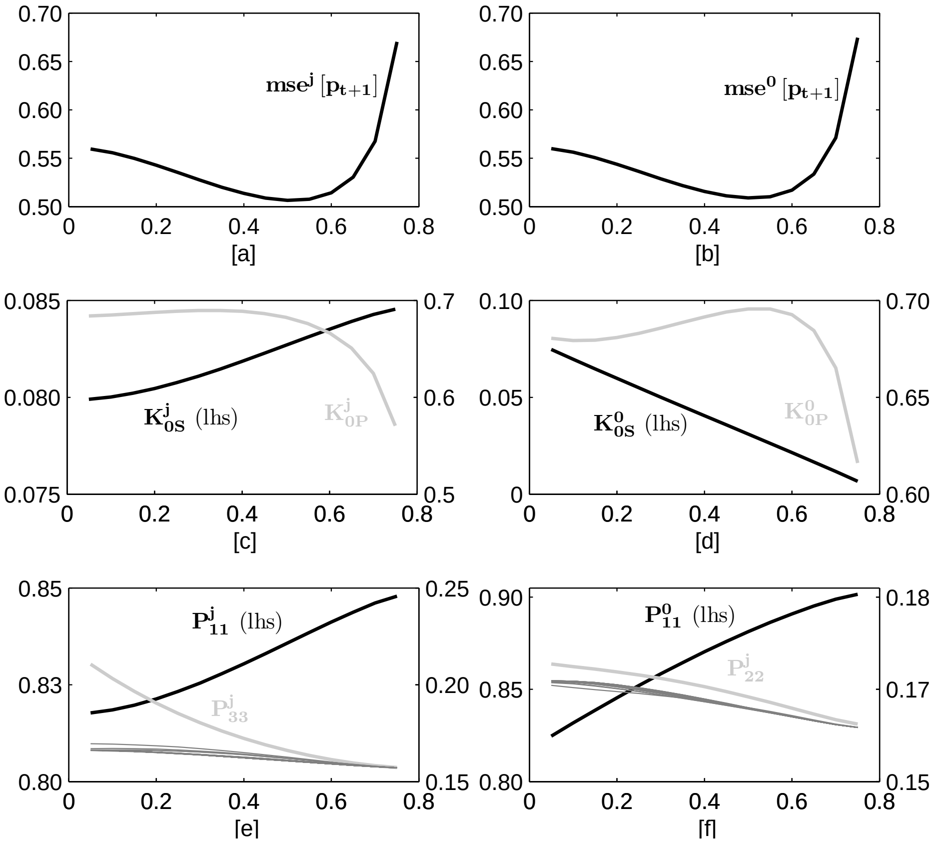
To illustrate this trade-off, Figure 6.1 demonstrates the effect of changes in m on the uncertainty about next period's price – the term relevant for welfare (see 3.2). In addition, I depict the effect on the mean-squared error of next-period's state-vector, X_{t+1}, as well as the Kalman Gains on s_t^i and p_t. Figure 6.2 shows a "compensating variation": the percentage change in the non-stochastic component of the coupon payment, $\bar{c}$, necessary to offset the effect on the utility of an individual trader of changes in uncertainty about next period's price. The baseline for Figure 6.2 is the welfare achieved at $m = 0.05$. All observations are relative to this base.

Panel (e) and (f) in Figure 6.1 shows that increases in the commonality of information increase uncertainty about the underlying fundamental for both types of traders, as we would expect. Uncertainty about *all* higher-order beliefs elements in the state vector, however, falls. The effect of knowing more about type zero traders' expectation, combined with their increased proportion in the population as we increase m, dominates knowing less about type $j \neq 0$. Uncertainty about *average* higher-order beliefs – the higher-order expectations that matter for next period's price – therefore falls. This is true for both traders of type $j \neq 0$ and for $i = 0$. Both types of traders benefit from a decrease in uncertainty about average higher-order beliefs, although type $j \neq 0$ naturally benefit slightly more.

The offsetting effect of knowing less about the fundamental but more about higher-order beliefs of that fundamental implies a u-shaped relationship: For low values of m, uncertainty about next period's price, and hence welfare, is decreasing in the mass of type zero traders – the benefit of knowing more about other traders' expectation, other traders' expectation of other traders' expectation, and so, outweigh the cost of knowing less about the fundamental. For high values of m, however, it is the other way around (Panel (a) and (b) in Figure 6.1). As such, increasing m from 0.05 to 0.50 implies a decrease in the non-stochastic component of the coupon payment necessary to keep welfare constant of -1.5% for type $j \neq 0$ and -1.0% for $i = 0$ (Figure 6.2). These numbers are, of course, merely descriptive, but it is nonetheless interesting that they are of reasonable economic significance.

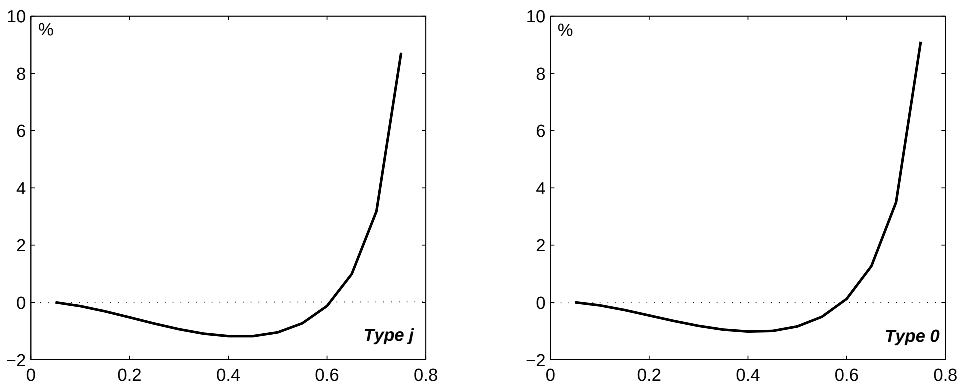
[15]The dynamic nature of our asset pricing model implies a (minor) offset to the above. The increase in the informativeness of the price about type z's expectation, for instance, implies that traders of type x update their beliefs by more for any given realization of the asset price. This paradoxically makes next-period higher-order expectations of type z's beliefs – those relevant for welfare – all else equal, harder to forecast. This type of offset is, however, numerically never very large.

Figure 6.1: Information Transmission and the Mass of Type Zero Traders



Information transmission as a function of $m = [0,\ 0.8]$. Panel (a and b) illustrate the one-period ahead mean-square error of the price forecast for agents of type $j \neq 0$ and $i = 0$ as function of the mass of type zero traders, respectively. Panel (c and d) show the associated weights placed on the type-specific signal and the market-clearing price for both segments of traders in their signal extraction of θ_t. Last, Panel (e and f) exhibit the mean-squared errors of one-period ahead state forecasts for agents of type $j \neq 0$ and $i = 0$, respectively.

Figure 6.2: Welfare and the Mass of Type Zero Traders



The chart illustrates as a function of $m = [0,\ 0.8]$ the percentage change in the coupon payment necessary to offset the effect on the utility of an individual trader of changes in uncertainty about next period's price. The baseline is the utility achieved at $m = 0$. All observations are relative to this base.

The responds of the Kalman Gains in Panel (c) and (f) in Figure 6.1 further illustrate the trade-off caused by changes in the commonality of information. The relative weight attached by type $j \neq 0$ traders to their private information when forming their beliefs about the fundamental is strictly increasing in m. This is due to the decrease in the informativeness of the price.[16] On the contrary, however, the relative weight attached by type $i = 0$ traders on their private information is strictly decreasing. Increases in the commonality of information imply a larger covariance, all else equal, between type zero traders' signals. The relatively smaller weight attached to their private information is because type zero traders attempt to compensate for their increased indirect reaction to ϵ_t^0 through the asset price (see Hellwig (1980)).[17] This further amplifies the decrease in uncertainty about type zero traders' beliefs.

6.2 Private Information and Higher-Order Beliefs

That asset prices communicate superior private information to less informed traders is well-established (see, for instance, Bernhardt et al. (2010)). By trading based on their superior private information well-informed traders partly reveal their informational advantage to the less-informed through changes in the asset price. This decrease in uncertainty should, all else equal, help the less-informed traders' better forecast future asset prices, increasing welfare. Compared to the literature, the approach outlined in Section 3 allows us to take a new look at how superior private information is conveyed through market-clearing prices by allowing us to analyze how higher-order expectations affect the diffusion of information.

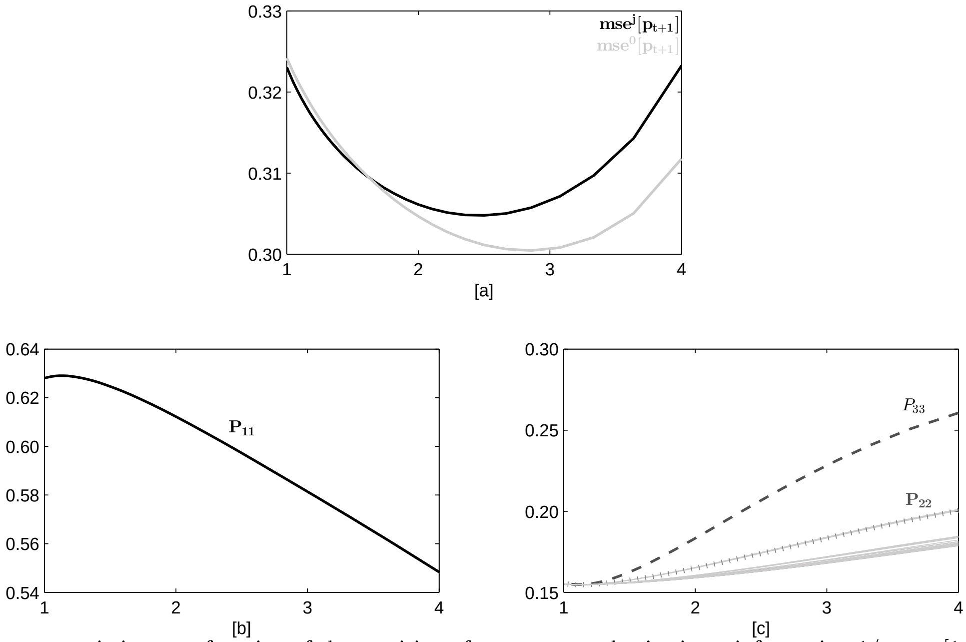
Proposition 4 and Corollary 2 show that the presence of higher-order expectations fundamentally affects the transmission of superior private information through market-clearing prices. Like with changes in the commonality of information, there is a trade-off between knowledge of the underlying fundamental and knowledge of other's beliefs of that fundamental. Providing superior private information to a segment of traders makes the asset price a better indicator of the underlying fundamental. This also decreases higher-order uncertainty about the less-informed traders' beliefs. But it also implies that the more informed traders will use their private information to a greater extent when forming their expectations. This, in turn, makes it harder for other traders to anticipate (future) changes in their beliefs. (And therefore also for the well-informed to predict changes in the less-informed's expectation of their beliefs, and so on.) This increases higher-order uncertainty, all else equal. Furthermore, if the beliefs of the well-informed traders matter sufficiently for the asset price that people are attempting to forecast, welfare can actually decrease for everyone – despite people knowing more about the true shocks hitting the economy. Welfare can decrease even for the traders with superior private information.

This result stands in contrast to "the conjecture" presented in Rondina and Walker (2012), suggesting that the formation of higher-order expectations always improves information diffusion, as it forces traders with superior information to use their private information to guess the forecast errors of others, imputing more of their information into the asset price. But, as demonstrated by Proposition 4 and Corollary 2, greater uncertainty about higher-order expectations can, potentially, outweigh the benefits from the

[16]Remember, these are Kalman Gains; the weights on both signals do not have to sum to a specific constant

[17]For very large values of m, the weight attached to the price actually decreases for both segments of traders (Panel c and d in Figure 6.1). This decline is caused by the "wisdom of the crowds" effect also causing, eventually, lower optimal weight on the asset price for type zero traders.

Figure 6.3: Information Transmission and the Accuracy of Private Signals



Information transmission as a function of the precision of type zero traders' private information $1/\sigma_{\epsilon^0} = [1/\sigma_\epsilon : 4/\sigma_\epsilon]$. Panel (a) illustrates the one-period ahead mean-squared error of the asset price; Panel (b and c) the mean-squared error of one-period ahead state forecasts for traders of type $j \neq 0$. $m = 1/2$.

increased encoding of information, increasing uncertainty about future prices.[18]

To demonstrate the trade-off between knowing more about the underlying fundamental but less about others expectation of the fundamental, Figure 6.3 plots uncertainty about the next period's price as a function of the relative accuracy of type zero traders' private information. In addition, I depict the uncertainty about the various elements of next-period's state vector for type $j \neq 0$ traders. The changes in uncertainty about the state vector for traders of type zero resemble closely those of type $j \neq 0$ and are therefore not illustrated.[19] Figure 6.4 shows the welfare consequences – once more expressed in terms of "compensating variation": how much in percentage you would have to change the non-stochastic component of the dividend process to offset the effect of the change in uncertainty on welfare.

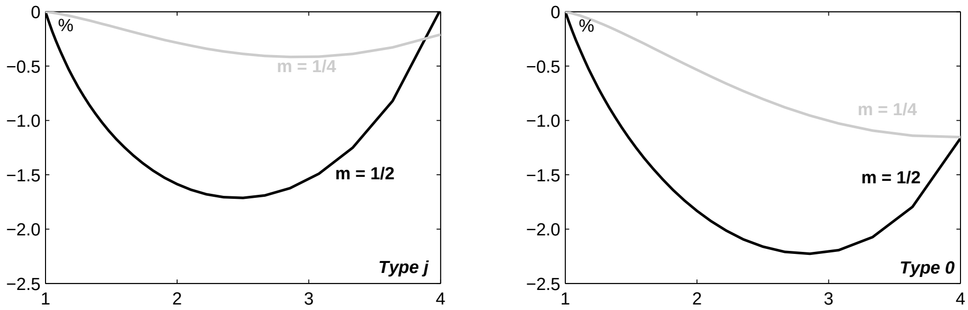
Figure 6.3 and 6.4 clearly show the role of higher-order beliefs in the dissemination of superior private information through market-clearing prices; we see another u-shaped relationship. Increasing the relative accuracy of type zero traders' private information from "the symmetric case" at first implies a decrease in uncertainty about next period's price – because of the rapid decrease in uncertainty about the underlying fundamental. But, for larger degrees of asymmetry of information, greater uncertainty about higher-order expectations eventually outweigh the benefits of knowing more about the underlying fundamental.[20] This is true for both types of traders. Uncertainty about next-period's price therefore

[18]The difference between the results presented in this paper and those of Rondina and Walker (2012) can be traced down to their rather "particular" information structure. In their model, the less-informed only observe the market-clearing price.

[19]As discussed in Section 4, even for $\sigma_{\epsilon^0} = \sigma_\epsilon$ there will be some asymmetry of information due to the correlation between type zero traders' two signals. In fact, only for $\sigma_{\epsilon^0} \ll \sigma_\epsilon$ will type zero traders be better at forecasting next period's price than type $j \neq 0$ traders.

[20]This also illustrates that the effect of knowing less about type zero traders' beliefs dominate knowing more about type

Figure 6.4: Welfare and the Accuracy of Private Signals



The chart illustrates as a function of $1/\sigma_{\epsilon^0} = [1/\sigma_\epsilon : 4/\sigma_\epsilon]$ the percentage change in the coupon payment necessary to offset the effect on the utility of an individual trader of changes in uncertainty about next period's price. The baseline is the utility achieved at $1/\sigma_{\epsilon^0} = 1/\sigma_\epsilon$. All observations are relative to this base.

at some moment becomes increasing in the relative accuracy of type zero traders' private information. More precise information decreases welfare for everyone in the population – *even* for the traders with superior private information.

Last, Figure 6.4 also demonstrates the crucial role of the parameter m, determining the relative size of type zero traders, for the communication of superior private information via asset prices. Decreasing the value of m, all else equal, implies smaller effects – both on the knowledge about the underlying fundamental and on higher-order beliefs – for both types of traders. As such, the welfare effects of changes in the relative accuracy of private information are only around $1/3$ of the size for type $j \neq 0$ when $m = 0.25$ as compared to $m = 0.50$. The welfare effects for type zero traders are similarly smaller, although this decrease is partially offset by a decrease in the correlation between type zero traders' two signals, decreasing their uncertainty about next period's price.

7 Conclusion

This paper asks how higher-order beliefs affect the information content of asset prices. I show that to understand the dissemination of information through market-clearing prices it is critical to recognize the role played by higher-order expectations. Specifically, propositions typically thought to decrease the "informational efficiency" of asset prices – such as increases in the commonality of information or decreases in the precision of private information – may, in fact, be welfare improving.

The simple insight behind this result is that what affects welfare is uncertainty about future prices; not uncertainty about the driving forces of the economy *per se*. With dispersed information, asset prices are in equilibrium a combination of both the underlying fundamentals of the economy and (higher-order) expectations of those fundamentals. This potentially creates a trade-off. Proposals like increases in the commonality of information or decreases in the precision of private information increase uncertainty

$j \neq 0$. Uncertainty about average higher-order beliefs are strictly increasing in the relative accuracy of type zero traders' private information.

about the driving forces but – importantly – decrease people's uncertainty about each other's beliefs. Future prices can therefore become easier to predict, improving welfare, despite everyone knowing less about the fundamentals affecting the price.

To illustrate the importance of this trade-off, I construct a basic dynamic asset pricing model. To solve it, I propose a solution technique for linear rational expectation models with dynamic higher-order expectations. Extending the work of Nimark (2011), I show that a truncated state-space solution can be applied to a class of linear models with asymmetric dispersed information and idiosyncratic private shocks affecting endogenous variables. The basic framework is very flexible, and can be used to analyze many economic situations in which there is asymmetric information and learning from endogenous variables.

While the asset pricing model used to demonstrate the potential for a trade-off between the different components of the asset price is highly stylized, techniques and insights similar to those developed here may prove useful for the study of how to best organize financial markets under imperfect dispersed information. This is left for future research

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