Appendix A: A Basic Model & The State-Space

A Basic Model

The Commonality of Information:

Proposition 1. *There is a unique equilibrium, where:*

$$y_1 = k_0\theta + k_1\epsilon_z$$

and k_0 and k_1 solve (2.5) and (2.6) in the main text, respectively. At the equilibrium solution, the informativeness of the public signal, y_1, about the fundamental, θ, is decreasing in the size of type z agents, m $\left(\frac{\partial k_1/k_0}{\partial m} > 0\right)$.

Proof. Equation (2.5) and (2.6) in the main text show that, for a unique equilibrium to exist, there has to be only one solution to the equation:

$$\frac{k_1}{k_0} = \frac{m}{m + (1-m)\, w_x} = \frac{m}{m + (1-m)\left[\frac{(k_1/k_0)^2}{1+2(k_1/k_0)^2}\right]}. \tag{1}$$

The left-hand side of this equation is the 45-degree line in $(k_1/k_0,\, k_1/k_0)$-space; the right-hand side is always within the interval $[\frac{m}{m+(1-m)\frac{1}{2}},\, 1]$ and strictly decreasing in $k_1/k_0 \geq 0$. A unique positive solution to this equation therefore exists for all values of $m \in [0,\, 1]$ (see Figure 1).

To show the second part of the proposition, note that an increase in m shifts up the entire right-hand side of the equation:

$$\frac{\partial}{\partial m}\left(\frac{m}{m + (1-m)\left[\frac{(k_1/k_0)^2}{1+2(k_1/k_0)^2}\right]}\right) = \frac{\frac{(k_1/k_0)^2}{1+2(k_1/k_0)^2}}{\left[m + (1-m)\frac{(k_1/k_0)^2}{1+2(k_1/k_0)^2}\right]^2} \geq 0.$$

The crossing with the 45-degree line therefore has to increase – k_1/k_0 has to, in equilibrium, go up. This, in turn, implies that the informativeness of the public signal, k_0/k_1, has to fall (see Figure 1). □

Proposition 2. *Agent x^i's uncertainty about the underlying fundamental, θ, equals:*

$$MSE_x^i(\theta) \equiv \mathbb{E}\left[\left(\theta - \mathbb{E}_x^i(\theta)\right)^2\right] = \frac{1}{\tau}\frac{1}{2 + (k_0/k_1)^2}.$$

Uncertainty about the underlying fundamental is therefore an increasing function of the share of the population comprised by agents of type z $\left(\partial MSE_x^i(\theta)/\partial m > 0\right)$. Uncertainty about type z agents' beliefs, however, is a decreasing function of m $\left(\partial MSE_x^i(\mathbb{E}_z\theta)/\partial m < 0\right)$, where

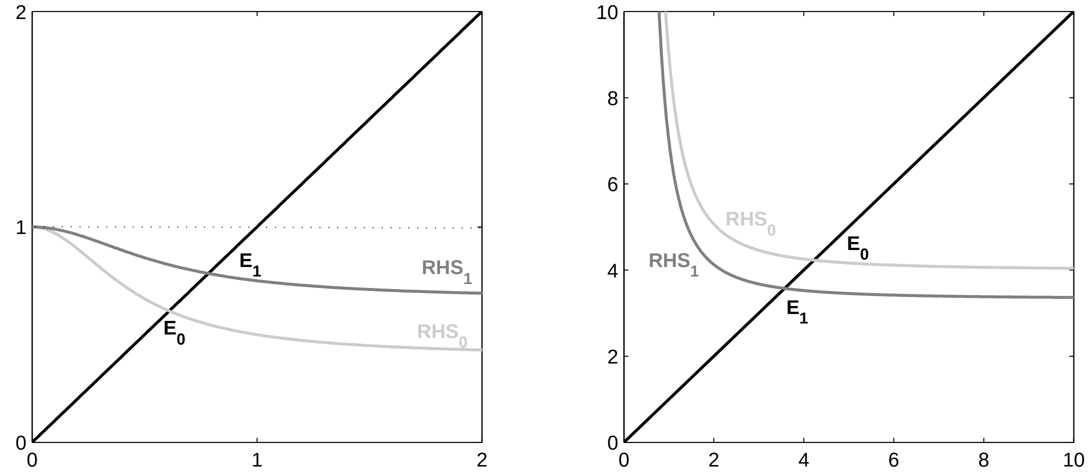
$$MSE_x^i(\mathbb{E}_z\theta) = \frac{1}{\tau}\frac{(1 - k_1/k_0)^2}{1 + 2(k_1/k_0)^2}.$$

Proof. Direct calculation shows that:

$$MSE_x^i(\theta) = \mathbb{E}\left[\left(\theta - \mathbb{E}_x^i(\theta)\right)^2\right] = \frac{1}{\tau}\frac{1}{2 + (k_0/k_1)^2},$$

where $\mathbb{E}_x^i[\theta]$ is defined in the body of this paper. Since $\frac{\partial(k_1/k_0)}{\partial m} > 0$ by Proposition 1, it immediately

Figure 1: Equilibrium Existence and Effects on k_1



RHS indicates the right-hand side of equation (1) [left chart] and (3) [right chart]. Both figures are drawn as a function of k_1. The left-hand side chart increases m from 0.25 to 0.50. The right-hand side chart increases τ_z from $\tau_z = \tau_x = \tau_\theta = 1$ to $\tau_z = 1.5$.

follows that $\frac{\partial MSE_x^i(\theta)}{\partial m} > 0$. This completes the first part of the proposition.

To show the second part of the proposition, note that:

$$\mathbb{E}_x^i \left[\mathbb{E}_z\left(\theta\right)\right] = \mathbb{E}_x^i\left[z\right] = \mathbb{E}_x^i\left[\theta\right] + \mathbb{E}_x^i\left[\epsilon_z\right]. \tag{2}$$

The first term on the right-hand side of (2) is derived in the main text. To derive the second term, combine the two signals x^i has to get:

$$\begin{aligned} x^i = \theta + \epsilon_x^i \quad &\rightarrow \quad \bar{x}^i \equiv \epsilon_z - \frac{k_0}{k_1}\epsilon_x^i \\ \tilde{y}_2 \equiv \theta + \frac{k_1}{k_0}\epsilon_z \quad &\qquad \bar{y}_2 \equiv \epsilon_z + \frac{k_0}{k_1}\theta. \end{aligned}$$

Then:

$$\mathbb{E}_x^i\left[\epsilon_z\right] = \bar{w}_x\bar{x}^i + \bar{w}_y\bar{y}, \quad \bar{w}_x \equiv \frac{\left(k_1/k_0\right)^2}{1 + 2\left(k_1/k_0\right)^2} \equiv \bar{w}_y.$$

Combining this expression with the one for $\mathbb{E}_x^i\left[\theta\right]$ in the main text gives us that:

$$\mathbb{E}_x^i\left[z\right] = w_x x^i + w_y\tilde{y} + \bar{w}_x\bar{x}^i + \bar{w}_y\bar{y}.$$

Direct calculation then shows us that:

$$MSE_x^i\left(z\right) = \mathbb{E}\left[\left(z - \mathbb{E}_x^i\left(z\right)\right)^2\right] = \frac{1}{\tau}\frac{\left(1 - k_1/k_0\right)^2}{1 + 2\left(k_1/k_0\right)^2}.$$

Since $\frac{\partial(k_1/k_0)}{\partial m} > 0$ by Proposition 1 and $0 \leq k_1/k_0 \leq 1$, the second part of the proposition follows $\left(\frac{\partial MSE_x^i(\mathbb{E}_z\theta)}{\partial m} < 0\right)$. □

The Precision of Private Information and Higher-Order Beliefs:

Proposition 3. *There is a unique equilibrium, where:*

$$y_2 = k_0\theta + k_1\epsilon_y$$

and k_0 and k_1 solve (2.10) and (2.11) in the main text, respectively. At the equilibrium solution, the informativeness of the public signal, y_2, about the fundamental, θ, is increasing in the precision of type z's private information, τ_z $\left(\frac{\partial k_1/k_0}{\partial \tau_z} < 0\right)$.

Proof. Equation (2.10) and (2.11) in the main text show that, for a unique equilibrium to exist, there has to be only one solution to the equation:

$$\frac{k_1}{k_0} = \frac{1}{w_z w_x} = \frac{\left[\tau_\theta + \tau_x + \left(\frac{k_0}{k_1}\right)^2 \tau_y\right]\left[\tau_\theta + \tau_z + \left(\frac{k_0}{k_1}\right)^2 \tau_y\right]}{\tau_z \tau_x}. \tag{3}$$

The left-hand side of this equation is the 45-degree line in $(k_1/k_0,\, k_1/k_0)$-space; the right-hand side is within the interval $\left[\frac{(\tau_\theta+\tau_x)(\tau_\theta+\tau_z)}{\tau_z\tau_x},\, \infty\right[$ and strictly decreasing in $k_1/k_0 \geq 0$. A unique positive solution exists (see Figure 1).

Differentiating the right-hand side of (3) with respect to τ_z shows that, for a given k_1/k_0, it is strictly decreasing. Increasing the precision of type z's private information must therefore, in equilibrium, lead to a decline in the ratio k_1/k_0 (see Figure 1). □

Proposition 4. *Agent x^i's uncertainty about the underlying fundamental, θ, equals:*

$$MSE_x^i(\theta) = \frac{1}{\tau_\theta + \tau_x + (k_0/k_1)^2 \tau_y}.$$

Uncertainty about θ is decreasing in the precision of type z's private information, τ_z $\left(\partial MSE_x^i(\theta)/\partial\tau_z < 0\right)$. Uncertainty about type z's expectation of θ can, however, be increasing in τ_z. In fact, for $\tau_\theta = \tau_x = \tau_y = \tau_z$:

$$\left.\frac{\partial MSE_x^i\left[\mathbb{E}_z(\theta)\right]}{\partial \tau_z}\right|_{\tau_z=\tau_\theta=\tau_y=\tau_x} > 0,$$

where

$$MSE_x^i\left[\mathbb{E}_z(\theta)\right] = \left(\frac{k_0}{k_1}\right)\left[\frac{1}{\tau_x} + \frac{1}{\tau_\theta + \tau_z + (k_0/k_1)^2\tau_y}\right].$$

Proof. Direct calculation shows that:

$$MSE_x^i(\theta) = \frac{1}{\tau_\theta + \tau_x + (k_0/k_1)^2 \tau_y}, \tag{4}$$

where $\frac{\partial k_1/k_0}{\partial\tau_z} < 0$ from Proposition 3. Thus $\frac{\partial MSE_x^i(\theta)}{\partial \tau_z} < 0$, showing the first part of the statement.

To show the second part, note that:

$$\mathbb{E}_x^i\left[\mathbb{E}_z(\theta)\right] = w_z\mathbb{E}_x^i[z] + w_y^z\tilde{y} = w_z\mathbb{E}_x^i[\theta] + w_y^z\tilde{y}$$

since $\tilde{y}$ is only a signal of θ and ϵ_y. Uncertainty about type z's expectation of θ therefore equals:

$$MSE_x^i\left[\mathbb{E}_z(\theta)\right] = w_z^2 MSE_x^i(\theta) + w_z^2\frac{1}{\tau_z}. \tag{5}$$

This illustrates the three effects on $MSE_x^i\left[\mathbb{E}_z(\theta)\right]$ discussed in the main text. Inserting (4) into (5) and canceling terms now gives us that:

$$MSE_x^i\left[\mathbb{E}_z(\theta)\right] = \left(\frac{k_0}{k_1}\right)^2\left[\tau_\theta + \tau_x + \left(\frac{k_0}{k_1}\right)^2\tau_y\right] + \frac{\tau_z}{\left[\tau_\theta + \tau_x + \left(\frac{k_0}{k_1}\right)^2\tau_y\right]^2}.$$

Combining this equation with the equilibrium relationship (3) implies that:

$$MSE_x^i\left[\mathbb{E}_z(\theta)\right] = \left(\frac{k_0}{k_1}\right)\left[\frac{1}{\tau_x} + \frac{1}{\tau_\theta + \tau_z + (k_0/k_1)^2\tau_y}\right].$$

We can now differentiate this expression with regards to τ_z evaluated at $\tau_\theta = \tau_x = \tau_y = \tau_z$ to get:

$$\left.\frac{\partial MSE_x^i\left[\mathbb{E}_z(\theta)\right]}{\partial\tau_z}\right|_{\tau_z=\tau_\theta=\tau_y=\tau_x} = \frac{1}{2+\kappa^2}\cdot \left[\frac{\partial\kappa}{\partial\tau_z}\left(2+\kappa^2\right)^2 + \kappa\left(2+\kappa^2\right)\left[1+\kappa^2-2\kappa\frac{\partial\kappa}{\partial\tau_z}\right] + \kappa^2 - 4\kappa\frac{\partial\kappa}{\partial\tau_z}\right], \quad (6)$$

where $\kappa \equiv k_0/k_1$. From (3) it follows that $\kappa < 1$. We likewise know that $\left[1+\kappa^2-2\kappa\frac{\partial\kappa}{\partial\tau_z}\right] > 0$ since the weight attached to the private signal z is increasing in its precision. Combining these two results with (6) shows that $\left.\frac{\partial MSE_x^i\left[\mathbb{E}_z(\theta)\right]}{\partial\tau_z}\right|_{\tau_z=\tau_\theta=\tau_y=\tau_x} > 0.$ □

The State Conjecture and The Infinite Regress:

To solve the model, given by equation (2.5), under dispersed asymmetric information, I conjecture (and will later verify) that the solution takes the following form:

$$\begin{aligned} p_t &= \tfrac{\beta\psi}{1-\beta\psi}c_t + \alpha' X_t^{(0:\infty)} + \lambda m\epsilon_t^0 + \lambda\xi_t \\ X_t^{(0:\infty)} &= MX_{t-1}^{(0:\infty)} + N\omega_t, \quad \omega_t = \left[\eta_t,\, \epsilon_t^0,\, \xi_t\right]', \end{aligned}$$

where $X_t^{(0:\infty)}$ denotes the relevant state vector. To solve for the state transition equation, we first though need to find the elements included in $X_t^{(0:\infty)}$. Proposition 5 provides those elements.

Proposition 5. *The state vector, $X_t^{(0:\infty)}$, for the asset pricing model is given by:*

$$X_t^{(0:\infty)} = \begin{bmatrix} X_t^{(0)} \\ X_t^{(1)} \\ \vdots \end{bmatrix}, \quad X_t^{(k)} = \begin{bmatrix} \bar{\mathbb{E}}_t X_t^{(k-1)} \\ \mathbb{E}_t^0 A_k X_t^{(k-1)} \end{bmatrix},$$

where A_k denotes an annihilation matrix that removes the last f_k rows from $\mathbb{E}_t^0\left[X_t^{(k-1)}\right]$ (see main text).

Proof. The proof is by induction (*sketch*). Assume that our conjecture for the state is:

$$\tilde{X}_t = \begin{bmatrix} \theta_t \\ \bar{\mathbb{E}}_t\theta_t \\ \mathbb{E}_t^0\theta_t \end{bmatrix} \equiv \begin{bmatrix} \theta_t \\ \theta_t^{(1)} \\ \theta_t^0 \end{bmatrix} = X_t^{(0:1)}.$$

To verify that $\tilde{X}_t$ follows a VAR(1), I now attempt to find expressions for $\theta_t^{(1)}$ and θ_t^0 as functions of

$\tilde{X}_{t-1}$ and ω_t. In terms of the state, type i's observation equation takes the form:

$$z_t^i \quad = \quad L\tilde{X}_t + Q_\epsilon \epsilon_t^i + Q_\omega \omega_t, \quad i \in \{0,\, 1,\, 2, \dots N\} \tag{7}$$

where the matrices are defined in the main text. Signal extraction is done using the Kalman Filter. Thus (see main text):

$$\begin{aligned} \bar{\mathbb{E}}_t\left[\tilde{X}_t\right] &= M\bar{\mathbb{E}}_{t-1}\left[\tilde{X}_{t-1}\right] + K\left(\bar{z}_t - LM\bar{\mathbb{E}}_{t-1}\left[\tilde{X}_{t-1}\right]\right) \\ &+ m\left(K^0 - K\right)\left(z_t^0 - LM\mathbb{E}_{t-1}^0\left[\tilde{X}_t\right]\right). \end{aligned} \tag{8}$$

While type zeros' expectation equals:

$$\mathbb{E}_t^0\left[\tilde{X}_t\right] = M\mathbb{E}_{t-1}^0\left[\tilde{X}_{t-1}\right] + K^0\left(z_t^0 - LM\mathbb{E}_{t-1}^0\left[\tilde{X}_t\right]\right). \tag{9}$$

z_t^0 and $\bar{z}_t$ can from (7) be expressed purely as functions of last periods' state, $\tilde{X}_{t-1}$, and the current period innovation, ω_t. In general, it is therefore *not* possible to find expressions for $\theta_t^{(1)}$ and θ_t^0 as functions of $\tilde{X}_{t-1}$ and ω_t alone. Our conjecture is consequently wrong. In fact, $\theta_t^{(1)} = e_1'\bar{\mathbb{E}}_t\left[\tilde{X}_t\right]$ and $\theta_t^0 = e_1'\mathbb{E}_t^0\left[\tilde{X}_t\right]$ will be functions of $\tilde{X}_{t-1}$, ω_t along with $\theta_{t-1}^{(2)}$, $\theta_{t-1}^{(1)0}$ and $\theta_{t-1}^{0(1)}$.

In sum, assuming only first order expectations of the underlying fundamental matter is wrong as these in general depend on second order expectations. Two potential solutions could exist: (i) Second order expectations, $\theta_{t-1}^{(2)}$, $\theta_{t-1}^{(1)0}$ and $\theta_{t-1}^{0(1)}$, might simply be a linear combination of $\tilde{X}_{t-1}$. But due to the Kalman Filtering this cannot be the case. For instance, from (8) we get that:

$$\theta_{t-1}^{(2)} = e_2'\bar{\mathbb{E}}_{t-1}\left[\tilde{X}_{t-1}\right] = f\left(\tilde{X}_{t-2},\, \bar{\mathbb{E}}_{t-2}\left[\tilde{X}_{t-2}\right],\, \mathbb{E}_{t-2}^0\left[\tilde{X}_{t-2}\right]\right),$$

where $f(\cdot)$ denotes some function, implying that even more past states matter. (ii) Include more states. We could try including $\theta_{t-1}^{(2)}$, $\theta_{t-1}^{(1)0}$ and $\theta_{t-1}^{0(1)}$ into our state-space. But, if we do this, and repeat the same exercise as above, we end up with third order expectations showing up, which implies an even larger state-space, and so on. More generally, from the above example, we can see that if we conjecture a state-space with k segments of higher-order expectations, the rational expectations equilibrium that arises using this conjecture implies that the relevant state-space has $k+1$ segments of higher-order expectations. By induction, the relevant state-space is of infinite order (see Towsend [1983]) and given by the expression in Proposition 5. □

Appendix B: The Solution Method

For ease of exposition, in this section, I consider the simplified asset pricing equation given by:

$$
\begin{aligned}
p_t = \beta \mathbb{E}_t [p_{t+1}] - \lambda\theta + \lambda\xi_t = \alpha' X_t + \lambda\xi_t \\
X_t = M X_{t-1} + N\omega_t.
\end{aligned}
\tag{10}
$$

Nothing of what follows depends on the exclusion of the coupon payments, c_t, or the systemically important traders idiosyncratic shock, ϵ_t^0.

Proposition 6. *[Nimark 2012a] For $|\beta\rho| < 1$ there exists, for each $\varepsilon > 0$, a $\bar{n} \in \mathbb{Z}_+$ such that*

$$\max \left|\alpha_{(n-1:n)}\right| < \varepsilon$$

for all $n \geq \bar{n}$.

Proof. See Nimark [2012a]. The difference in the degree of signal precision between the agents does not change that if $\theta_t = X_t(i) = 1 \forall i \in \mathbb{Z}_+$– i.e. if all expectations in the hierarchy are equal to the actual value of the fundamental – then common knowledge of rational expectations implies that all expectations about future values of θ_t must coincide with first-order expectations. Therefore, forwarding (10) gives:

$$
\begin{aligned}
p_t &= -\lambda \sum_{j=0}^{\infty} e_1' \left[\beta M \begin{bmatrix} \underline{0} & I & \underline{0} \end{bmatrix} H^{-1}\right]^j X_t + \lambda\xi_t \\
&= -\lambda \sum_{j=0}^{\infty} e_1' \left[\beta M \begin{bmatrix} \underline{0} & I & \underline{0} \end{bmatrix} H^{-1}\right]^j \mathbf{1} + \lambda\xi_t \\
&= \alpha' \mathbf{1} + \lambda\xi_t,
\end{aligned}
$$

where we can note that, as in Nimark [2012a]:

$$e_1' \left[\beta M \begin{bmatrix} \underline{0} & I & \underline{0} \end{bmatrix} H^{-1}\right]^j \mathbf{1} = (\beta\rho)^j .$$

□

Lemma 1. *Let $\overline{\mathbb{E}}_t [X_t(i)]$ denote the average expectation of element i in X_t at time t, and let $\mathbb{E}_t^j [X_t(i)]$ denote the corresponding expectation for agents of type $j \neq 0$, then if $|\rho| < 1$ and $\mathbb{V}\left[\mathbb{E}_t^0 [X_t(i)]\right] \leq \mathbb{V}\left[\mathbb{E}_t^j [X_t(i)]\right]$:*

$$\mathbb{V}\left[\overline{\mathbb{E}}_t [X_t(i)]\right] \leq \mathbb{V}\left[\mathbb{E}_t^j [X_t(i)]\right].$$

But if instead $\mathbb{V}\left[\mathbb{E}_t^0 [X_t(i)]\right] \geq \mathbb{V}\left[\mathbb{E}_t^j [X_t(i)]\right]$, then

$$\mathbb{V}\left[\overline{\mathbb{E}}_t [X_t(i)]\right] \leq \mathbb{V}\left[\mathbb{E}_t^0 [X_t(i)]\right].$$

Proof. By definition:

$$\overline{\mathbb{E}}_t\left[X_t(i)\right] \equiv \sum_{h=0}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\phi(h) = m\mathbb{E}_t^0\left[X_t(i)\right] + (1-m)\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]. \tag{11}$$

Assume: $\mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right] \leq \mathbb{V}\left[\mathbb{E}_t^j\left[X_t(i)\right]\right]$. Using the MA form of agents' expectations, we get that:

$$\begin{aligned}\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right] &= \sum_{h=1}^{N} \phi_1(L)\eta_t + \phi_2(L)\xi_t + \phi_3(L)\epsilon_t^h \\ &= \phi_1(L)\eta_t + \phi_2(L)\xi_t,\end{aligned}$$

implying that also $\mathbb{V}\left[\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\right] \leq \mathbb{V}\left[\mathbb{E}_t^j\left[X_t(i)\right]\right]$. Now taking the variance of (11) gives:

$$\begin{aligned}\mathbb{V}\left[\overline{\mathbb{E}}_t\left[X_t(i)\right]\right] &= m^2\mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right] + (1-m)^2\mathbb{V}\left[\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\right] \\ &+ 2m(1-m)cov\left\{\mathbb{E}_t^0\left[X_t(i)\right], \sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\right\} \\ &\leq m^2\mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right] + (1-m)^2\mathbb{V}\left[\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\right] \\ &+ 2m(1-m)\max\left\{\mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right], \mathbb{V}\left[\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\right]\right\},\end{aligned} \tag{12}$$

where I have used Cauchy-Schwarz' Inequality in L^2.
If $\mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right] \geq \mathbb{V}\left[\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\right]$, this expression collapses to:

$$\begin{aligned}\mathbb{V}\left[\overline{\mathbb{E}}_t\left[X_t(i)\right]\right] &\leq \left[1+m^2-2m\right]\mathbb{V}\left[\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\right] + \left[2m-m^2\right]\mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right] \\ &\leq \mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right] \leq \mathbb{V}\left[\mathbb{E}_t^j\left[X_t(i)\right]\right].\end{aligned}$$

Similarly, *if* $\mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right] \leq \mathbb{V}\left[\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\right]$, (12) becomes:

$$\begin{aligned}\mathbb{V}\left[\overline{\mathbb{E}}_t\left[X_t(i)\right]\right] &\leq m^2\mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right] + \left[1-m^2\right]\mathbb{V}\left[\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\right] \\ &\leq \mathbb{V}\left[\mathbb{E}_t^j\left[X_t(i)\right]\right].\end{aligned}$$

Now assume instead: $\mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right] \geq \mathbb{V}\left[\mathbb{E}_t^j\left[X_t(i)\right]\right]$. It follows that $\mathbb{V}\left[\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\right] \leq \mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right]$. Therefore, we can re-write equation (12) as:

$$\begin{aligned}\mathbb{V}\left[\overline{\mathbb{E}}_t\left[X_t(i)\right]\right] &\leq \left[1+m^2-2m\right]\mathbb{V}\left[\sum_{h=1}^{N} \mathbb{E}_t^h\left[X_t(i)\right]\right] + \left[2m-m^2\right]\mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right] \\ &\leq \mathbb{V}\left[\mathbb{E}_t^j\left[X_t(i)\right]\right] \leq \mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right].\end{aligned}$$

□

Proposition 7. *The kth order of the hierarchy is given by:*

$$X_t^{(k)} = \begin{bmatrix} \bar{\mathbb{E}}_t X_t^{(k-1)} \\ \mathbb{E}_t^0 A_k X_t^{(k-1)} \end{bmatrix}, \quad k \geq 1.$$

If $|\rho| < 1$*, then:*

$$\begin{aligned} diag\left\{\mathbb{V}\left[\bar{\mathbb{E}}_t X_t^{(k-1)}\right]\right\} &\leq diag\left\{\mathbb{V}\left[X_t^{(k-1)}\right]\right\} \\ diag\left\{\mathbb{V}\left[\mathbb{E}_t^0 A_k X_t^{(k-1)}\right]\right\} &\leq diag\left\{\mathbb{V}\left[A_k X_t^{(k-1)}\right]\right\}, \end{aligned}$$

where $diag\{\cdot\}$ *denotes the diagonal elements of a matrix.*

Proof. The (k+1)th order of the hierarchy is comprised of three components: [1] the average expectation of elements in the kth order that are average expectations; [2] the average expectation of elements in the kth order that are of type zeros expectation; and lastly [3] type zeros expectation of average expectations in the kth order.

Beginning with elements of type [1], linear projection gives us that:

$$\bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right] = \bar{w}_y y + \bar{w}_p p_t + \bar{w}_\theta \theta_t + \bar{w}_{s^0} s_t^0, \tag{13}$$

where, for ease of exposition, I have denoted the prior y, treating it as a constant in the below. Also, due to symmetry, throughout this proof I will assume that $\mathbb{V}\left[\mathbb{E}_t^0\left[X_t(i)\right]\right] \leq \mathbb{V}\left[\mathbb{E}_t^j\left[X_t(i)\right]\right]$. These assumptions merely simplify the exposition, without changing the result. i indexes the elements in the (k-1)th order. Agents of type $j \neq 0$ expectation of (13) is given by:

$$\begin{aligned} \mathbb{E}_t^j \bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right] &= \bar{w}_y y + \bar{w}_p p_t + \bar{w}_\theta \mathbb{E}_t^j \theta_t + \bar{w}_{s^0} \mathbb{E}_t^j s_t^0 \\ &= \bar{w}_y y + \bar{w}_p p_t + \bar{w}_\theta \mathbb{E}_t^j \theta_t + \bar{w}_{s^0} \mathbb{E}_t^j \theta_t \\ &= \bar{w}_y y + \bar{w}_p p_t + \left(\bar{w}_\theta + \bar{w}_{s^0}\right)\left(\theta_t - e_t^j\right), \end{aligned} \tag{14}$$

where I have used that $s_t^0 = \theta_t + \epsilon_t^0$, $\epsilon_t^0 \sim WN\left(0, \sigma_{\epsilon^0}^2\right)$, and that from the Projection Theorem: $\theta_t = \mathbb{E}_t^j \theta_t + e_t^j$, where e_t^j is orthogonal to $\mathbb{E}_t^j \theta_t \in \Omega_t^j$.
The variance of (14) is:

$$\begin{aligned} \mathbb{V}\left\{\mathbb{E}_t^j \bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right]\right\} + \left(\bar{w}_\theta + \bar{w}_{s^0}\right)^2 \mathbb{V}\left[e_t^j\right] &= \bar{w}_p^2 \mathbb{V}\left[p_t\right] + \left(\bar{w}_\theta + \bar{w}_{s^0}\right)^2 \mathbb{V}\left[\theta_t\right] + \\ &+ 2\bar{w}_p\left(\bar{w}_\theta + \bar{w}_{s^0}\right) cov\left(p_t, \theta_t\right) \\ &= \mathbb{V}\left\{\bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right]\right\} \\ &- \bar{w}_{s^0}^2 \mathbb{V}\left[\epsilon_t^0\right] - 2\bar{w}_p \bar{w}_{s^0} cov\left(p_t, \epsilon_t^0\right). \end{aligned}$$

Since all terms are positive, it follows that:

$$\mathbb{V}\left\{\mathbb{E}_t^j \bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right]\right\} \leq \mathbb{V}\left\{\bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right]\right\}.$$

And using Lemma 1:

$$\mathbb{V}\left\{\bar{\mathbb{E}}_t\bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right]\right\}\leq\mathbb{V}\left\{\bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right]\right\}.$$

Continuing with elements of type [2], linear projection gives us that:

$$\mathbb{E}_t^0\left[\left(A_{k-1}X_t^{(k-1)}\right)(l)\right]=w_y^0y+w_p^0p_t+w_{s^0}^0s_t^0, \tag{15}$$

where l indexes the elements in the (k-1)th order that are average expectations.
Agents of type $j\neq 0$ expectation of (15) is given by:

$$\begin{aligned}\mathbb{E}_t^j\mathbb{E}_t^0\left[\left(A_{k-1}X_t^{(k-1)}\right)(l)\right] &= w_y^0y+w_p^0p_t+w_{s^0}^0\mathbb{E}_t^js_t^0\\ &= w_y^0y+w_p^0p_t+w_{s^0}^0\mathbb{E}_t^j\theta_t\\ &= w_y^0y+w_p^0p_t+w_{s^0}^0\left(\theta_t-e_t^j\right),\end{aligned} \tag{16}$$

The variance of (16) is:

$$\begin{aligned}\mathbb{V}\left\{\mathbb{E}_t^j\mathbb{E}_t^0\left[\left(A_{k-1}X_t^{(k-1)}\right)(l)\right]\right\}+\left(w_{s^0}^0\right)^2\mathbb{V}\left[e_t^j\right] &= \left(w_p^0\right)^2\mathbb{V}\left[p_t\right]+\left(w_{s^0}^0\right)^2\mathbb{V}\left[\theta_t\right]+\\ &+ 2w_p^0w_{s^0}^0cov\left(p_t,\,\theta_t\right)\\ &= \mathbb{V}\left\{\mathbb{E}_t^0\left[\left(A_{k-1}X_t^{(k-1)}\right)(l)\right]\right\}\\ &- \left(w_{s^0}^0\right)^2\mathbb{V}\left[\epsilon_t^0\right]-2w_p^0w_{s^0}^0cov\left(p_t,\,\epsilon_t^0\right).\end{aligned}$$

Since all terms are (again) positive, we get, using Lemma 1:

$$\mathbb{V}\left\{\bar{\mathbb{E}}_t\mathbb{E}_t^0\left[\left(A_{k-1}X_t^{(k-1)}\right)(l)\right]\right\}\leq\mathbb{V}\left\{\mathbb{E}_t^0\left[\left(A_{k-1}X_t^{(k-1)}\right)(l)\right]\right\}.$$

Lastly, considering elements of type [3], agents of type zeros expectation of average expectations in the kth order are, using (13):

$$\begin{aligned}\mathbb{E}_t^0\bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right] &= \bar{w}_yy+\bar{w}_pp_t+\bar{w}_\theta\mathbb{E}_t^0\theta_t+\bar{w}_{s^0}\mathbb{E}_t^0s_t^0\\ &= \bar{w}_yy+\bar{w}_pp_t+\bar{w}_\theta\left(\theta_t-e_t^0\right)+\bar{w}_{s^0}s_t^0.\end{aligned}$$

Taking the variance of this expression gives:

$$\mathbb{V}\left\{\mathbb{E}_t^0\bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right]\right\}+\bar{w}_\theta^2\mathbb{V}\left[e_t^0\right]=\mathbb{V}\left\{\bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right]\right\}.$$

Thus:

$$\mathbb{V}\left\{\mathbb{E}_t^0\bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right]\right\}\leq\mathbb{V}\left\{\bar{\mathbb{E}}_t\left[X_t^{(k-1)}(i)\right]\right\}.$$

Finally, combining our results from the three cases, we get that:

$$\begin{aligned}diag\left\{\mathbb{V}\left[\bar{\mathbb{E}}_tX_t^{(k)}\right]\right\}&\leq diag\left\{\mathbb{V}\left[X_t^{(k)}\right]\right\}\\ diag\left\{\mathbb{V}\left[\mathbb{E}_t^0A_kX_t^{(k)}\right]\right\}&\leq diag\left\{\mathbb{V}\left[A_kX_t^{(k)}\right]\right\}.\end{aligned}$$

□

Proposition 8. *If $\beta < 1$ and θ_t is stationary, the variance of the approximation error, $\mathbb{V}\left[p_t - p_t^k\right]$, where $p_t^k \equiv \alpha'_{(0:k)} X_t^{(0:k)} + \lambda\xi_t$, tends to zero as $k \to \infty$.*

Proof. To show this, I will first show that the sequence:

$$\alpha'_{(0:k)} \mathbb{V}\left[X_t^{(0:k)}\right] \alpha_{(0:k)} \geq 0$$

is bounded[1]. Note that:

$$\begin{aligned} \alpha'_{(0:k)} \mathbb{V}\left[X_t^{(0:k)}\right] \alpha_{(0:k)} &= \sum_{i=0}^{k} \sum_{j=0}^{f_i} \sum_{h=0}^{k} \sum_{l=0}^{f_h} \alpha_{i,j} \alpha_{l,h} cov\left(X_t(i,j),\, X_t(l,h)\right) \\ &\leq \sum_{i=0}^{k} \sum_{j=0}^{f_i} \sum_{h=0}^{k} \sum_{l=0}^{f_h} \alpha_{i,j} \alpha_{l,h} \max\left(\mathbb{V}\left[X_t(i,j)\right], \mathbb{V}\left[X_t(l,h)\right]\right) \end{aligned} \quad (17)$$

where f_i denotes the number of elements in the ith order, $\alpha_{i,j}$ the jth element of the ith order in α and $X_t(i,j)$ the corresponding element in X_t. I have also (again) applied Cauchy-Schwarz' Inequality in L^2. From Proposition 6, we know that as k goes to infinity, the elements of $\alpha_{k,j}$ go to zero. In addition, from Proposition 7, we know that the variance of higher order expectations is a (piecewise) non-increasing sequence.[2] Thus, from (17), we can conclude that $\alpha'_{(0:k)} \mathbb{V}\left[X_t^{(0:k)}\right] \alpha_{(0:k)}$ is a bounded sequence. Also, since $\alpha'_{(0:k)} \mathbb{V}\left[X_t^{(0:k)}\right] \alpha_{(0:k)}$ is a bounded sequence with a limit given by $\alpha' \mathbb{V}\left[X_t\right] \alpha$, we know that the limit is also finite.

The variance of the approximation error is given by:

$$\mathbb{V}\left[p_t - p_t^k\right] = \alpha'_{(k+1:\infty)} \mathbb{V}\left[X_t^{(k+1:\infty)}\right] \alpha_{(k+1:\infty)}.$$

Therefore:

$$\mathbb{V}\left[p_t - p_t^k\right] + \alpha'_{(0:k)} \mathbb{V}\left[X_t^{(0:k)}\right] \alpha_{(0:k)} + 2\alpha'_{(k+1:\infty)} cov\left(X_t^{(0:k)},\, X_t^{(k+1:\infty)}\right) \alpha_{(0:k)} = \alpha' \mathbb{V}\left[X_t\right] \alpha,$$

implying that $\mathbb{V}\left[p_t - p_t^k\right] \to 0$ as $k \to \infty$ if $2\alpha'_{(k+1:\infty)} cov\left(X_t^{(k+1:\infty)},\, X_t^{(0:k)}\right) \alpha_{(0:k)} \to 0$ as $k \to \infty$. But, as:

$$2\alpha'_{(k+1:\infty)} cov\left(X_t^{(k+1:\infty)},\, X_t^{(0:k)}\right) \alpha_{(0:k)} = \sum_{h=k+1}^{\infty} \sum_{l=0}^{f_h} \sum_{i=0}^{k} \sum_{j=0}^{f_i} \alpha_{i,j} \alpha_{l,h} cov\left(X_t(i,j),\, X_t(l,h)\right),$$

Proposition 6 and 7 imply (for the same reason as just above) that this term converges to zero as k increases. □

Remark 1. The variance of the approximation error, $\mathbb{V}\left[p_t - p_t^k\right]$, where $p_t^k \equiv \alpha'_{(0:k)} X_t^{(0:k)} + \lambda\xi_t$, tends to zero as $k \to \infty$ iff. $\mathbb{V}\left[p_t\right]$ is finite.

[1] If this sequence was not always greater than or equal to zero, we would be able to choose a discount rate and a variance of ξ_t such that the variance of p_t^k would be less than zero.

[2] "Piecewise" here refers to the fact that we have shown in Proposition 7 that: $diag\left\{\mathbb{V}\left[\bar{\mathbb{E}}_t X_t^{(k)}\right]\right\} \leq diag\left\{\mathbb{V}\left[X_t^{(k)}\right]\right\}$ and $diag\left\{\mathbb{V}\left[\mathbb{E}_t^0 A_k X_t^{(k)}\right]\right\} \leq diag\left\{\mathbb{V}\left[A_k X_t^{(k)}\right]\right\}$.

Proof. To see that if the proposed truncation method works, the variance of the price is finite, note that since:

$$\mathbb{V}\left[p_t - p_t^k\right] = \mathbb{V}\left[p_t\right] + \mathbb{V}\left[p_t^k\right] - 2cov\left(p_t,\, p_t^k\right) \to 0,$$

all elements of the sequence have to be bounded. If:

$$\mathbb{V}\left[p_t\right] = 2cov\left(p_t,\, p_t^k\right) - \mathbb{V}\left[p_t^k\right] \quad \forall k,$$

the variance of price can still be infinite. But this would imply that:

$$\mathbb{V}\left[p_t - p_t^k\right] = 0 \quad \forall k,$$

which is a contradiction.

To see that if the variance of the price is finite, then the variance of the approximation error tends to zero, consider:

$$\mathbb{V}\left[p_t\right] = \alpha'\mathbb{V}\left[X_t\right]\alpha + \lambda^2\mathbb{V}\left[\xi_t\right] + 2\alpha'\lambda cov(X_t, \xi_t).$$

If the variance of the price is finite, all terms have to be finite, including $\alpha'\mathbb{V}\left[X_t\right]\alpha$. Thus, following the last steps in Proposition 8:

$$\mathbb{V}\left[p_t - p_t^k\right] = \left[\alpha' - \left[\begin{array}{cc} \alpha'_{(0:k)} & \underline{0} \end{array}\right]\right]\mathbb{V}\left[X_t\right]\left[\alpha' - \left[\begin{array}{cc} \alpha'_{(0:k)} & \underline{0} \end{array}\right]\right]' \to 0.$$

□

Proposition 9. *[Nimark 2012a] The mapping* $\left\{M_j^{\bar{k}},\, N_j^{\bar{k}}\right\} \to \left\{M_{j+1}^{\bar{k}},\, N_{j+1}^{\bar{k}}\right\}$ *is continuous and from a convex compact set onto itself. Brouwer's Fix-point Theorem therefore applies, guaranteeing the existence of a fix-point.*

Proof. As the variance of an optimal estimate can never be larger than the variance of the object being estimated, Proposition 7 must hold for each iteration j. Thus, repeatedly applying Proposition 7, using Cauchy-Schwartz' Inequality in L^2, implies that:

$$\mathbb{V}_j\left[X_t^{(0:k)}\right]_{ih} \leq \mathbb{V}\left[\theta_t\right],$$

where $\mathbb{V}_j\left[\cdot\right]$ denotes the variance conditional on the evolution of the state being described by $M_j^{\bar{k}}$ and $N_j^{\bar{k}}$, and i and h indexes the elements in the variance matrix. Similarly, Proposition 4 implies that:

$$cov_j\left(X_t^{(0:k)},\, X_{t+1}^{(0:k)}\right)_{ih} \leq \max\left\{\mathbb{V}_j\left[X_t^{(0:k)}(i)\right],\, \mathbb{V}_j\left[X_{t+1}^{(0:k)}(h)\right]\right\} \leq \mathbb{V}\left[\theta_t\right],$$

where i and h here index the elements in the covariance matrix.

From linear projection, we know that:

$$\begin{aligned} \mathbb{E}\left[X_{t+1}^{(0:k)} \mid X_t^{(0:k)}\right] &= M^{\bar{k}} X_t^{(0:k)} \\ &= cov_j\left(X_t^{(0:k)},\, X_{t+1}^{(0:k)}\right)\mathbb{V}_j\left[X_t^{(0:k)}\right]^{-1}. \end{aligned}$$

Now following the exact same steps as in Nimark (2012a) shows the result. □

Appendix C: Dynamics

This Appendix considers the effect of asymmetric dispersed information on the dynamics of the market-clearing price. I first analyze how the dynamics of the asset price change due to the impact of type zero traders' idiosyncratic shock on the endogenous variable. I then investigate how asymmetric information diffuses through the asset price, and how the dissemination of this information affects the dynamics of the endogenous variable. As I will show, higher-order expectations play a crucial role in understanding the effect of asymmetric dispersed information. To demonstrate the various effects, I maintain the parametrization used in the main text.

Asset-Pricing Dynamics with Dispersed Asymmetric Information

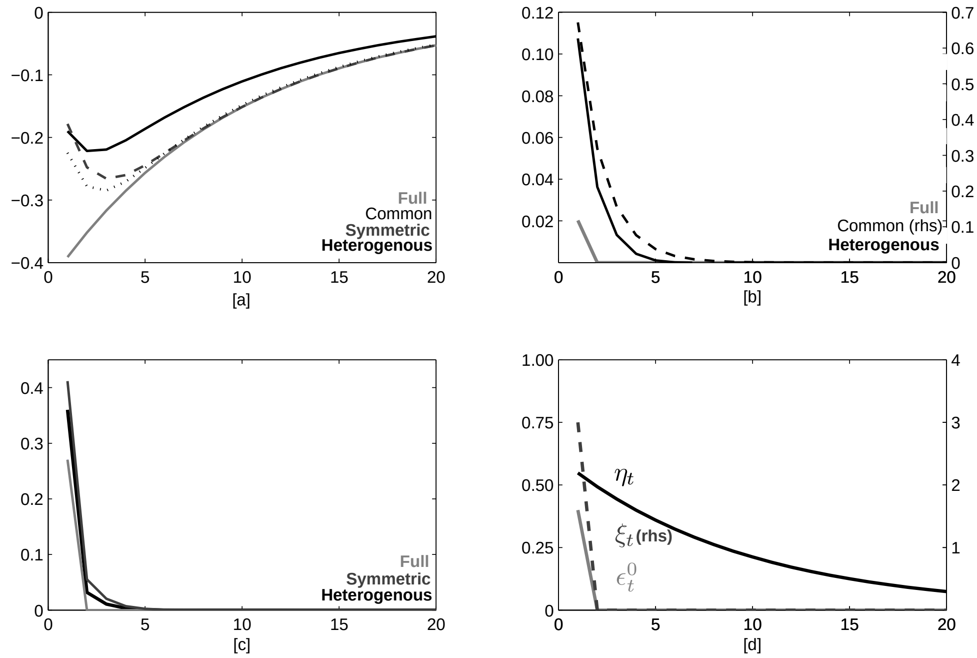
To take an initial look at how asymmetric information (caused by the impact of type zero traders' idiosyncratic noise shock on the endogenous variable) changes the dynamics of the asset price, Figure 2 considers the impulse response of p_t to the three aggregate shocks: the innovation to the persistent supply fundamental, η_t; the innovation to the idiosyncratic supply noise, ξ_t; and lastly, the noise shock to systemically important traders' type-specific signal, ϵ_t^0. For comparison purposes, I have included the impulse response of the market-clearing price under complete information; imperfect-but-common information, where each agent observes type zero traders' signal; and lastly, symmetric dispersed information, where all agents observe the asset price as well as a private signal that has no bearing on p_t. Figure 3 helps clarify the response of p_t under asymmetric dispersed information by showing the associated impulse response of the expectation hierarchy, $X_t^{(0:\bar{k})}$, split into average expectations, $\bar{\mathbb{E}}_t \left[X_t^{(0:\bar{k})}\right]$, and type zero expectations, $\mathbb{E}_t^0 \left[X_t^{(0:\bar{k})}\right]$.

Panel (a) in Figure 2 illustrates the impact of a shock to the underlying fundamental, θ_t, on the asset price. Imperfect information blunts the initial response of the market-clearing price to the persistent supply innovation when compared to the full information solution. Moreover, the trough response occurs later, and is more persistent, across all imperfect information cases. Contrasting the two private information responses with that from common information (with an equally accurate common signal), we see that private information can generate substantial additional inertia in the response of p_t. As argued in, for instance, Woodford (2002), the cause of the added inertia from private information is that – in response to shocks that have a purely private information component – higher-order expectations are more sluggish to react than lower order expectations (see Panel (a, b) in Figure 3). This in turn implies a more inertial response of the endogenous variable.[3]

In addition to the above well-established results, Figure 2 also illustrates the impact of heterogeneous signal precision, caused by the impact of systemically important traders' type-specific shock on p_t, on the dynamics of the asset price. Comparing the response under symmetric dispersed information with the asymmetric case, we note that the heterogeneity in the signal accuracy causes substantially more persistence in the impulse response. Panel (a, b) in Figure 3 explains why: In response to an innovation in θ_t, average expectations (of average expectations, and so on) are less persistent than expectations of type zero agents. The reason for this discrepancy is that – under our parameter configuration – traders of type zero are relatively more confounded by whether any change in the price is a result of a change in their idiosyncratic liquidity shock, or the underlying fundamental, because of the stronger correlation between their signals. Optimally, agents of type $i = 0$ therefore update their expectations more slowly,

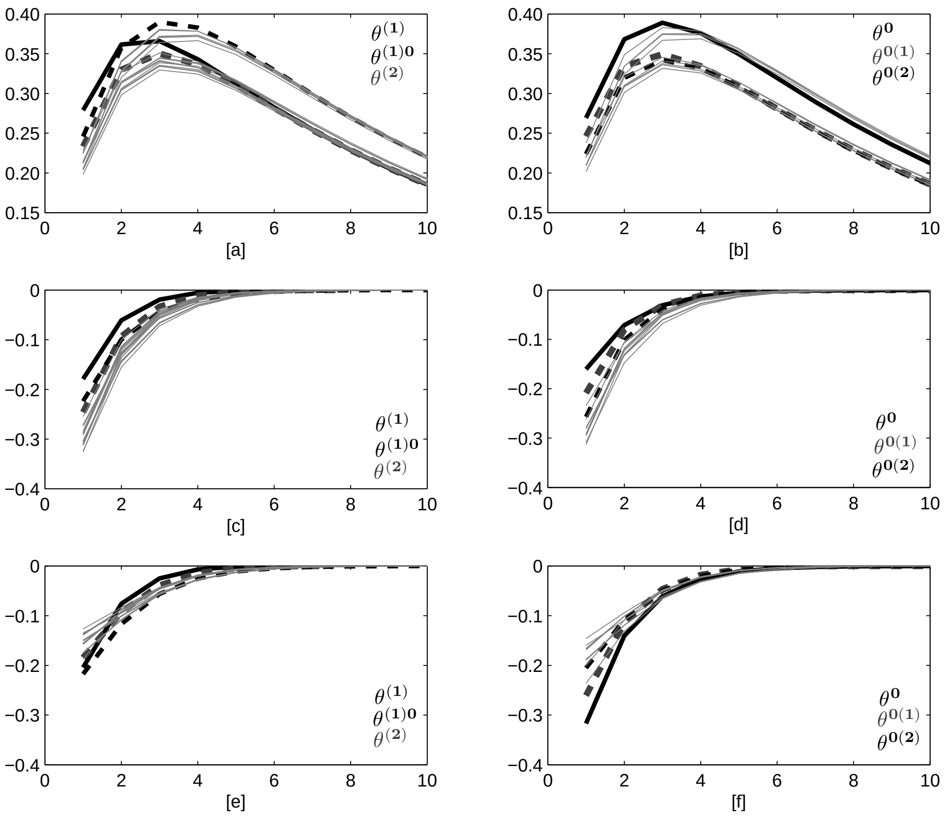
[3]The reason why higher order-expectation responses are less responsive than lower order is due to the increased uncertainty about the higher-order expectation as we move down in the hierarchy (see Woodford (2002) for a thorough discussion).

Figure 2: Impulse Response of Market-Clearing Price



Impulse response of p_t to a one-standard deviation shock to the underlying fundamental, η_t, the idiosyncratic supply noise, ξ_t, and the liquidity shock to type zero agents, ϵ_t^0, respectively. Four cases are considered: (i) full information, (ii) imperfect-but-common information, (iii) symmetric dispersed information, and lastly (iv) heterogeneous dispersed information. Panel (a) shows the impulse response to η_t, Panel (b) the impulse response to ϵ_t^0 and Panel (c) the impulse response to ξ_t Last, Panel (d) demonstrates the time series behavior of the shocks.

Figure 3: Impulse Response of $X_t^{(0:\bar{k})}$



Impulse response of $X_t^{(0:\bar{k})}$ to a one-standard deviation shock to the underlying fundamental, η_t, the idiosyncratic supply noise, ξ_t, and the liquidity shock to type zero agents, ϵ_t^0, respectively. Panel (a), (c) and (e) show the impulse response to *average expectations,* $\bar{\mathbb{E}}_t\left[X_t^{(0:\bar{k})}\right]$, while Panel (b), (c) and (d) exhibit the impulse response to type zero agents expectations, $\mathbb{E}_t^0\left[X_t^{(0:\bar{k})}\right]$. Row one demonstrates the impact of η_t; row two the impact of ξ_t; and lastly, row three the impact of ϵ_t^0. For readability of the chart, only the responses to the first six orders of expectations are included, and only the first three components of each hierarchy are denoted.

implying a more persistent response of higher-order expectations. The greater persistence in the price response under asymmetric dispersed information is thus caused by type zero agents lower average signal precision (which is again caused by the impact of their type-specific shock on the endogenous variable).

Panel (c) in Figure 2 shows the impact of a shock to the idiosyncratic supply innovation, ξ_t, on the market-clearing price. By definition the response in the imperfect-but-common and full information case are the same. That said, both dispersed information cases respond significantly more on impact than the full information solution. In addition, their impulse responses are also more persistent. Increases in the (noisy) price lead the agents under dispersed information to partially attribute the increase in the price to a decrease in θ_t, implying both a stronger, and more persistent, price response. Moreover, as the price is a public signal (implying that everyone knows, that everyone knows, and so on, that each other observe the price), the results in Allen et al. (2006) apply. That is, in response to an innovation to a public signal, higher-order expectations respond stronger than lower order expectations, imparting both extra persistence and amplification in the response of p_t to ξ_t under dispersed information.[4]

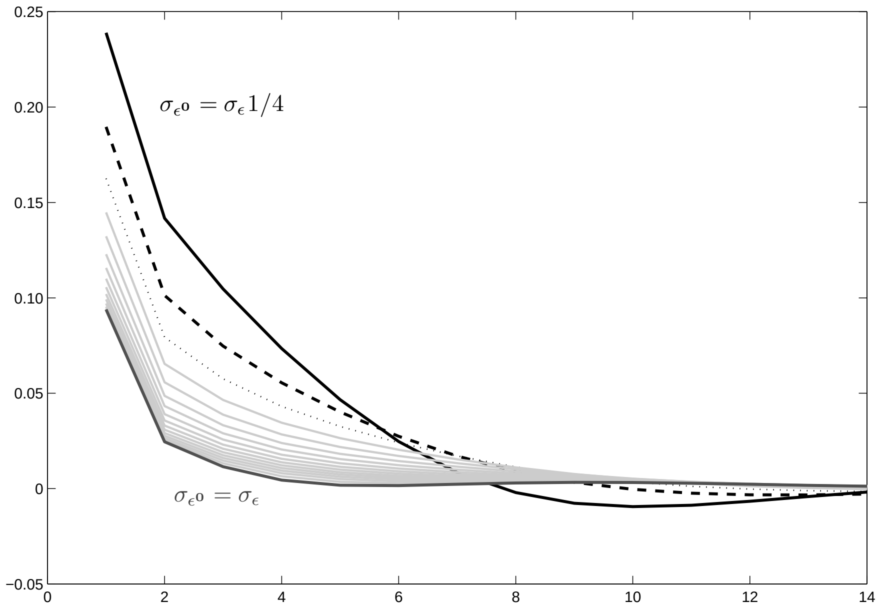
The impact of heterogeneous signal precision can again be assessed by comparing, in Panel (c), the response under symmetric dispersed information with the asymmetric counterpart. Similar to before, the symmetric dispersed information case responds more to the ξ_t shock than the asymmetric. Because of the stronger correlation between their signals, type $i = 0$ traders place a relatively smaller weight across the orders of expectation on the public signal as they attempt to compensate for their indirect reaction to ϵ_t^0 through the price. The decreased weight also explains why higher-order expectations increase less rapidly for type zero traders in Panel (c and d) in Figure 3 compared to average expectations.

Last, Panel (b) in Figure 2 illustrates the impact of a shock to ϵ_t^0 on p_t. By assumption there is no impact on the symmetric dispersed information solution. In contrast, the impact on the imperfect-but-common information case is, clearly, the largest. All agents, under this information structure, receive a private signal of a persistent decrease in θ_t, increasing demand along with the asset price. Under dispersed asymmetric information, however, only a share m of the total mass of agents receive a type-specific signal, indicating a persistent decrease in θ_t. That said, all agents notice an increase in the market-clearing price, likewise indicating a fall in the underlying fundamental. As in Panel (a and c), part of the reason for the enduring response of the endogenous variable is the persistent response of higher-order expectations (Panel (e, f) in Figure 3). But unlike with the innovation to ξ_t, the shock to ϵ_t^0 is partially an innovation to a public signal, and part-wise an innovation to a type-specific signal. Higher-order *average* expectations thus do not uniformly respond more vigorously than lower order expectations. Moreover, for type $i = 0$, innovations to ϵ_t^0 have similar effects to changes in θ_t, implying that the increasing effect of a shock to the public signal on higher-order expectations is offset by the decreasing effect of a shock to the type-specific signal – exactly in the same way as in Panel (a) (Panel (f) in Figure 3). The impact on both their signals, naturally, amplifies the reaction of type zero agents' expectations, when compared to the impact on type $j \neq 0$ traders.

In sum, responses of higher-order expectations to innovations in ϵ_t^0 correspond for type $j \neq 0$ closely to innovations in the *idiosyncratic* supply fundamental, while for type $i = 0$ they resemble changes to the *persistent* supply fundamental. Average higher-order expectations will react according to the linear combination of these two responses.

[4]That higher-order expectations respond stronger than lower order is intuitive (see Panel (c, d) in Figure 3): When forecasting the value of the underlying fundamental, agents use optimally both their signals. But when forecasting, for instance, the average forecast of the fundamental, the knowledge that all other agents have also observed the price, causes the individual agent to optimally attach more weight to the public signal, as it is a relatively better indicator of the average forecast than the type-specific signal.

Figure 4: Impulse Response of p_t to ϵ_t^0 for different values of σ_{ϵ^0}



The impulse response of p_t to a one unit shock to ϵ_t^0 for values of $\sigma_{\epsilon^0} = [\sigma_\epsilon/4 : 0.01 : \sigma_\epsilon]$.

Asset-Pricing Dynamics and the Asymmetry of Information

This Subsection explores the effect of changes in the asymmetry of information on the dynamics of the market-clearing price. I start by analyzing the impact of shocks to type zero traders' private signal, as I vary its precision. I then investigate how the trade-off, identified in the main text, between knowing more about the fundamental but less about the knowledge of others' affects the dynamics of the market-clearing price. As I show, the role of higher-order expectations in the transmission of information is critical for understanding how varying degrees of signal precision alter the dynamics of asset prices.

The Responsiveness of p_t to ϵ_t^0

Figure 4 illustrates the effect of ϵ_t^0 on the market-clearing price as we vary the precision in the systemically important agents' signal from $\sigma_{\epsilon^0} = \sigma_\epsilon$ to $\sigma_{\epsilon^0}^j < \sigma_\epsilon$, $j = 1, 2, ..., 16$, where $\sigma_{\epsilon^0}^{j+1} < \sigma_{\epsilon^0}^j$. Increasing levels of private signal precision for type zero traders imply a significantly larger, and more persistent impact, of the noise in the systemically important types' signal on the market-clearing price – even for relatively small values of m. Type-specific noise thus moves the price substantially when there is large amounts of asymmetric information. Cutler et al. (1989) document that many of the biggest movements in the S&P500 have occurred without any (ex-post) important public information about the underlying fundamental being revealed on the day. Figure 4 argues that such movements could correspond to optimal responses to noise in systemically important traders' private information during periods of large information asymmetry.

The reason for the increasing responsiveness of p_t to ϵ_t^0 as we vary σ_{ϵ^0} is two-fold: First, there is a direct effect stemming from any noise shock to $i = 0$ traders type-specific information mattering more for their beliefs about θ_t the higher the signal-to-noise ratio in s_t^0 is. Any noise shock to type $i = 0$ traders will therefore imply larger changes in the asset price. And second, there is an indirect effect caused by the increasing weight placed on the price by agents of type $j \neq 0$.[5] Private information becomes imperfectly revealed to the other market participants via the market-clearing price. When traders of type $i = 0$ receive, and use in their trading decisions, on average more precise private information, more accurate information about θ_t becomes encoded in market-clearing price. Traders of type $j \neq 0$ therefore optimally place a larger weight on the endogenous variable in their signal-extraction as they attempt to learn from this new, more accurate, private information. But the larger weight on the public signal also implies that any error in traders of type $i = 0's$ type-specific signal of θ_t has a larger impact on the beliefs of type $j \neq 0$, and thus on the asset price. The transmission of improved private information via the endogenous variable therefore restores some of the complementarity in action that is lost by the systemically important traders attaching more weight to s_t^0 in their signal extraction.

Higher-Order Expectations and the Dynamics of the Asset Price

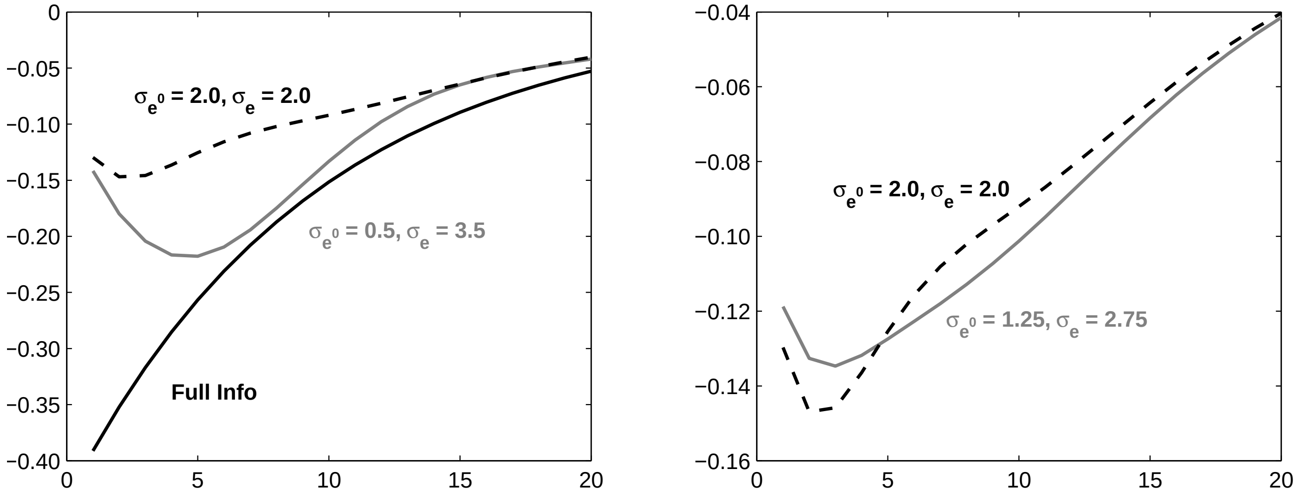
The role of higher-order expectations for the transmission of information through prices, discussed in the main text, helps explain how asymmetric information alters the dynamics of the asset price. Below, I show the effect of a "symmetric increase" in the amount of asymmetric information across agents on the dynamics of p_t.[6] The encoding of superior private information in the market-clearing price, all else equal, implies that a situation with more asymmetric dispersed information more closely resembles the outcome under full information (see left-hand side chart in Figure 5). However, as agents become increasingly unsure about each others expectations, the price may, for certain parameter values, respond less on impact than the symmetric information counterpart, thus resembling less the full information solution (see right-hand side chart in Figure 5). Which of these two offsetting effects dominate therefore also determines whether asymmetric dispersed information causes initial responses to the fundamental that are closer or further away from the full information case.

Figure 6 provides an interesting example, where the increased encoding of superior information always causes the market-clearing price under asymmetric dispersed information to initially more closely resemble the full information solution. But as agents become increasingly uncertain about higher-order expectations, they also update their expectations more slowly, imparting more persistence in the dynamic response. In Figure 6, this increased persistence implies that the price after ten periods can be well below what it would be under full information.

[5] In the words of Wang (1994), type $j \neq 0$ agents increasingly become "price chasers".

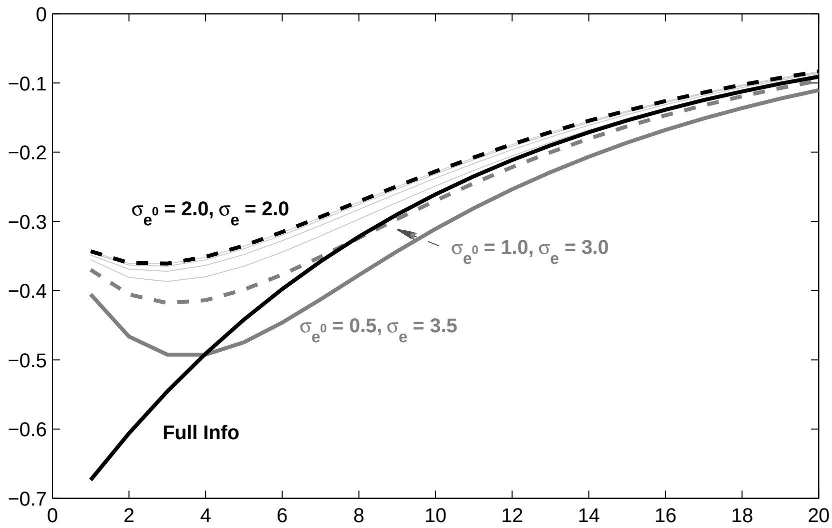
[6] Figure 4 dealt with the impact of systemically important traders type-specific shock on p_t for the dynamics of the asset price. In contrast, I here look at the impact of "symmetric" changes in the signal accuracy of s_t^i across agents on the dynamics of the market-clearing price.

Figure 5: Asymmetric Information and the Dynamics of p_t



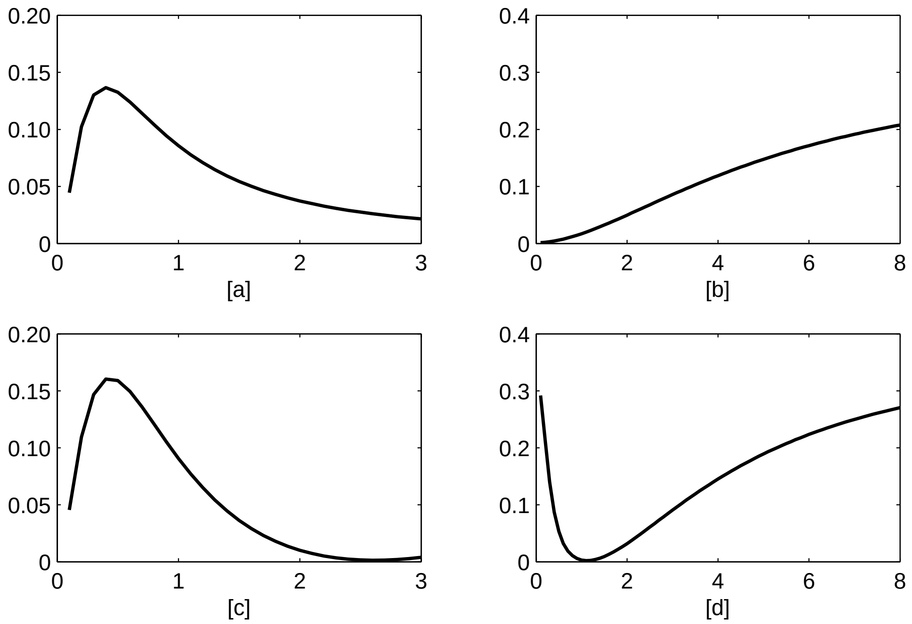
Impulse response of p_t to a one unit shock to η_t. For both the left- and the right-hand chart, $m = 0.5$. The charts illustrate the impact of a symmetric change in the signal accuracy of s_t^i across agents on the dynamics of the asset price.

Figure 6: Asymmetric Information and the Dynamics of p_t: an " Excess Response"



Impulse response of p_t to a one unit shock to η_t. The size of type zero traders is $m = 0.5$. To decrease the importance of higher-order thinking, I set $\delta = 0.5$. The chart illustrates the impact of a symmetric change in the signal accuracy of s_t^i across agents on the dynamics of the asset price.

Figure 7: Cross-Sectional Dispersion of Price Expectations



Panel (a, b) show the cross-sectional dispersion of one-period ahead price expectations for type $j \neq 0$ agents for different values of σ_ϵ^2 and σ_ξ^2. Panel (c, d), in contrast, illustrate the cross-sectional dispersion of one-period ahead price expectations *across* type $i = 0$ and $j \neq 0$, again for different values of σ_ϵ^2 and σ_ξ^2. Appendix D details the underlying calculations.

Appendix D: Cross-Sectional Dispersion

The previous Appendix showed that the extent to which the price dynamics are different under asymmetric dispersed information compared to the symmetric information counterpart depends on the degree of dispersion between type $i = 0$ and type $j \neq 0$ traders' expectation of the state. The dispersion between these two sets of agents will again depend on the degree of asymmetric information as well as the information transmission that happens through the endogenous price. In our model, asymmetric information can be caused by differences in signal precision of type-specific information, but also by the correlation between the two elements of type $i = 0$ traders' signal vector. For the baseline parameter values considered in this subsection, however, only the latter effect exists. This subsection therefore illustrates the dispersion of beliefs in the baseline setup between type $i = 0$ and type $j \neq 0$ traders' expectation as a function of the standard deviation of the noise in the type-specific signal, $\sigma_\epsilon = \sigma_{\epsilon^0}$, as well as the standard deviation of the idiosyncratic supply innovation, σ_ξ. In the body of this paper I break the $\sigma_\epsilon = \sigma_{\epsilon^0}$ assumption and consider in more general terms the effect of asymmetric information on the informativeness and dynamics of the market-clearing price. In addition to the dispersion of beliefs between type $i = 0$ and $j \neq 0$ agents, dispersed private information also causes the beliefs within type $j \neq 0$ to differ. Figure 7 accordingly illustrates both the *within* type $j \neq 0$ and *across* types $i = 0$ and $j \neq 0$ dispersion of one-period ahead price expectations.

The findings in the top row of Figure 7 replicate the results in Nimark (2012): First, there is a hump-shaped relationship between σ_ϵ and the within type $j \neq 0$ dispersion. And second, there is no such trade-off for changes in σ_ξ. As within type $j \neq 0$ dispersion is caused by differences in type-specific information, increasing the variance of the public signal, implies uniformly higher weight on the private information, and hence always more cross-sectional variation.

The difference in beliefs between type $i = 0$ and type $j \neq 0$ traders is depicted in the second row of Figure 7. Similar to Panel (a), Panel (c) shows a hump-shaped relationship between σ_ϵ and the across type dispersion – and for very much the same reason: As the type-specific signal becomes very precise, there is little doubt about the true value of θ_t, implying little across type variation in the one-period ahead price forecast. Similarly, for very imprecise type-specific information, agents of both type $i = 0$ and of type $j \neq 0$ attach almost all of the weight in their signal extraction on the public signal, suggesting also little cross-sectional variation. Only for intermediate values of σ_ϵ does the stronger correlation between type $i = 0$ traders signals imply meaningfully different Kalman Gain weights, imparting a large cross-sectional dispersion in price forecasts.

Contrary to Panel (b), the across-type variation in Panel (d) is not a continuously increasing function. The reason is due to the two offsetting effects, discussed in Section 4, determining whether the systemically important type is better or worse informed about the underlying fundamental. As $\sigma_\xi \to 0$, an increasing amount of the "noise" in the price is caused by the idiosyncratic liquidity shock to type zero agents. The systemically important agents can thus increasingly use their type-specific signal – with the eventually lower signal-to-noise ratio – as an indicator of the "noise" in the price, relying on the public signal, and the strong correlation between their two signals, to inform them about θ_t. Optimally, agents of type zero therefore attach a *positive and increasing* weight on s_t^0 for low and decreasing values σ_ξ, implying a large across-type dispersion in one-period ahead price forecasts (type $j \neq 0$ agents attach increasingly small negative weights to their type-specific signals). In contrast, as σ_ξ increases, the price becomes a worse signal of the underlying fundamental. Eventually, the declining signal-to-noise ratio in the public signal, and the corresponding declining correlation between their two indicators, cause traders of type zero to attach a *negative* weight to their private signal, as they attempt to learn about the underlying fundamental from their type-specific information. As σ_ξ increases even further, the relative weight on s_t^0 correspondingly increases, causing more across-type variation in price forecasts. Clearly, in the limiting case where $\sigma_\xi \to \infty$, asymmetric information disappears under our baseline parameter values, and the price dynamics converge to the symmetric information counterpart. Below I detail the calculations underlying Figure 7.

Within Type $j \neq 0$ One-period ahead Price Dispersion:

The *within* type $j \neq 0$ one-period ahead price dispersion is defined as the expected value of the difference between the next period price forecast for type j and the average one-period ahead price forecast across all $j \neq 0$ types squared. In other words as:

$$\mathbb{E}\left\{\left[\mathbb{E}_t^j\left[p_{t+1}\right] - \sum_{h=1}^N \mathbb{E}_t^h\left[p_{t+1}\right]\right]^2\right\} = \alpha' M \mathbb{E}\left\{\left[\mathbb{E}_t^j\left[X_t\right] - \sum_{h=1}^N \mathbb{E}_t^h\left[X_t\right]\right]\left[\mathbb{E}_t^j\left[X_t\right] - \sum_{h=1}^N \mathbb{E}_t^h\left[X_t\right]\right]'\right\} M'\alpha. \tag{18}$$

To find the dispersion in traders of type $j \neq 0$ estimate of the state, I use that:

$$\mathbb{E}_t^j\left[X_t\right] = \left[I - KL\right] M \mathbb{E}_{t-1}^j\left[X_{t-1}\right] + K z_t^j.$$

Hence:

$$\mathbb{E}\left\{\left[\mathbb{E}_t^j[X_t]-\sum_{h=1}^{N}\mathbb{E}_t^h[X_t]\right]\left[\mathbb{E}_t^j[X_t]-\sum_{h=1}^{N}\mathbb{E}_t^h[X_t]\right]'\right\} \equiv$$
$$\Omega^j = [I-KL]\,M\Omega^j M'\,[I-KL]'+\sigma_\epsilon^2 KQ_\epsilon Q_\epsilon' K'.$$

The solution to this Riccatti equation gives the dispersion in traders of type $j\neq 0$ estimate of the state. Equation (18) then gives the corresponding one-period ahead price expectation.

Across Type $i=0$ and $j\neq 0$ One-period ahead Price Dispersion:

The across type dispersion of one-period ahead price expectations is given by:

$$\mathbb{E}\left\{\left[\left(\mathbb{E}_t^0[p_{t+1}]-\mathbb{E}_t^j[p_{t+1}]\right)-\mathbb{E}\left(\mathbb{E}_t^0[p_{t+1}]-\mathbb{E}_t^j[p_{t+1}]\right)\right]^2\right\} =$$
$$\alpha' M\mathbb{V}\left[\mathbb{E}_t^0[X_t]-\mathbb{E}_t^j[X_t]\right]M'\alpha, \quad (19)$$

To find the variance of the difference in the state estimate between type $i=0$ and type $j\neq 0$, note that:

$$\begin{aligned}\mathbb{V}\left[\mathbb{E}_t^0[X_t]-\mathbb{E}_t^j[X_t]\right] &= \mathbb{V}\left[\mathbb{E}_t^0[X_t]\right]+\mathbb{V}\left[\mathbb{E}_t^j[X_t]\right]\\ &- \mathbb{E}\left[\left(\mathbb{E}_t^0X_t-\mathbb{E}\mathbb{E}_t^0X_t\right)\left(\mathbb{E}_t^jX_t-\mathbb{E}\mathbb{E}_t^jX_t\right)'\right]-\mathbb{E}\left[\left(\mathbb{E}_t^0X_t-\mathbb{E}\mathbb{E}_t^0X_t\right)\left(\mathbb{E}_t^jX_t-\mathbb{E}\mathbb{E}_t^jX_t\right)'\right]'.\end{aligned}$$

The variance terms in this expression are given by the solution to the Riccatti equations:

$$\begin{aligned}\mathbb{V}\left[\mathbb{E}_t^0[X_t]\right] &= \left[I-K^0L\right]M\mathbb{V}\left[\mathbb{E}_t^0[X_t]\right]M'\left[I-K^0L\right]'\\ &+ \left(K^0LM\right)\mathbb{V}[X_t]\left(K^0LM\right)'+K^0\left(Q_0+LN\right)\Sigma\left(Q_0+LN\right)'K^0\end{aligned}$$

and

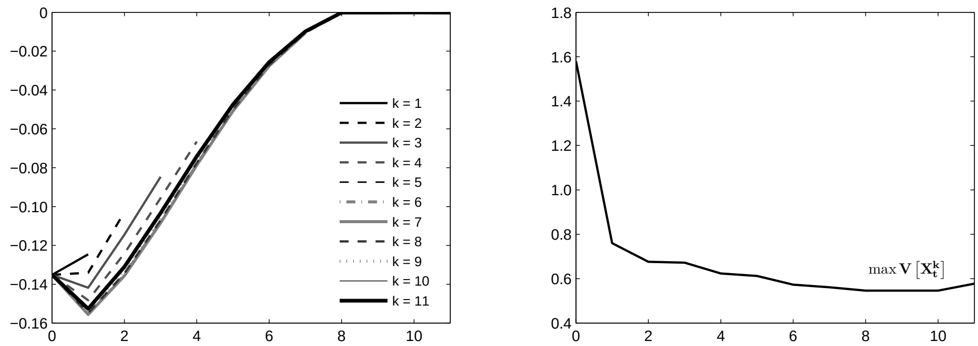
$$\begin{aligned}\mathbb{V}\left[\mathbb{E}_t^j[X_t]\right] &= [I-KL]\,M\mathbb{V}\left[\mathbb{E}_t^j[X_t]\right]M'[I-KL]'\\ &+ (KLM)\,\mathbb{V}[X_t]\,(KLM)'+K\left(Q_\omega+LN\right)\Sigma\left(Q_\omega+LN\right)'K.\end{aligned}$$

Similarly, the covariance terms are given by the solution to another Riccatti equation:

$$\begin{aligned}cov\left\{\mathbb{E}_t^j[X_t],\mathbb{E}_t^0[X_t]\right\} &= [I-KL]\,Mcov\left\{\mathbb{E}_t^j[X_t],\mathbb{E}_t^0[X_t]\right\}M'\left[I-K^0L\right]'\\ &+ K\left[LM\mathbb{V}[X_t]\,M'L'+(LN+Q_\omega)\,\Sigma\,(LN+Q_0)'\right].\end{aligned}$$

Inserting the solution to these three Riccatti equations into (19) gives the across type dispersion of one-period ahead price expectations.

Figure 8: Price Coefficients and The Variance of the Hierarchy



The left-hand side chart illustrates the min $\{\alpha_k\}$ for each k and for all $\bar{k} = \{1, 2, \ldots, 11\}$. The right-hand side charts shows max $\left\{diag\left\{\mathbb{V}\left[X_t^{(k)}\right]\right\}\right\}$ as a function of k.

Appendix E: Accuracy of Solution

To operationalize the solution algorithm developed above and in the main text, we need to choose a maximum for the order of expectations $\bar{k}$. This Appendix illustrates two approaches how to choose $\bar{k}$, in addition to demonstrating that the key propositions above are indeed satisfied for the given parameter setup used in the main text. The left-hand side chart in Figure 8 shows the minimum coefficient in α for each order of expectation for $\bar{k} = \{1, 2, \ldots 11\}$. As Figure 8 illustrates, the min $\left\{\alpha_{(\bar{k}-1:\bar{k})}\right\}$ converges surprising rapidly to zero. Already at $\bar{k} = 8$, the minimum element of an extra order is close to zero. In addition, the coefficients on *all* previous orders appear to have stabilized. All else equal, Figure 8 therefore indicates that a relatively low order of expectation can be used to capture the dynamics of the market-clearing price. That said, as highlighted in the proof of Proposition 7, the rapid converges of min $\left\{\alpha_{(\bar{k}-1:\bar{k})}\right\}$ is not sufficient to guarantee that the variance of the approximation error converges to zero. However, as the right-hand side chart in Figure 8 shows, the maximum variance in each order of expectation of the hierarchy also stabilizes at around $\bar{k} = 8$.[7] Thus, orders of expectation above around seven to eight do not seem to add meaningfully to the dynamics of $p_t^{\bar{k}}$, allowing for a relatively parsimonious state-space to approximate the dynamics of the asset price, p_t.

Another, potentially more direct, approach to choose $\bar{k}$, is to measure changes in impulse responses to aggregate shocks. Figure 9 shows the norm of the change in the impulse response function for $\bar{k} = \{2, 3, \ldots 11\}$. As in the previous charts, Figure 9 demonstrates that relatively low orders of expectation can capture the dynamics of the asset price: At $\bar{k} = 10$, there is no discernible change in the impulse response to any of the aggregate shocks.

The exact value of $\bar{k}$ needed to accurately approximate the dynamics of the asset price, of course, depends on the exact parameter configuration used. Generally, the results in this paper show that the more persistent the underlying fundamental is, the more higher-order expectations are all else equal necessary. In addition, the impact of higher-order expectations is maximized for intermediate levels of signal precision. For very precise signals, higher-order dynamic expectations are accurately captured

[7] The small increase in the maximum element of $diag\left\{\mathbb{V}\left[X_t^{(k)}\right]\right\}$ that you can note for high levels of k is not due to a mistake in Proposition 7. Rather it is an artifact of the truncation based approach. Higher-order expectations, on average, respond negatively to lagged higher, higher-order expectations (as these "overreact" to public information). Truncating the state-space therefore implies that these negative coefficients in the M matrix are not "all" included for $X_t^{(k)}$. Thus, the calculated variance for $X_t^{(k)}$, depicted in Figure 4, is higher than the actual variance – and mostly so for large values of k.

Figure 9: Changes in Impulse Responses

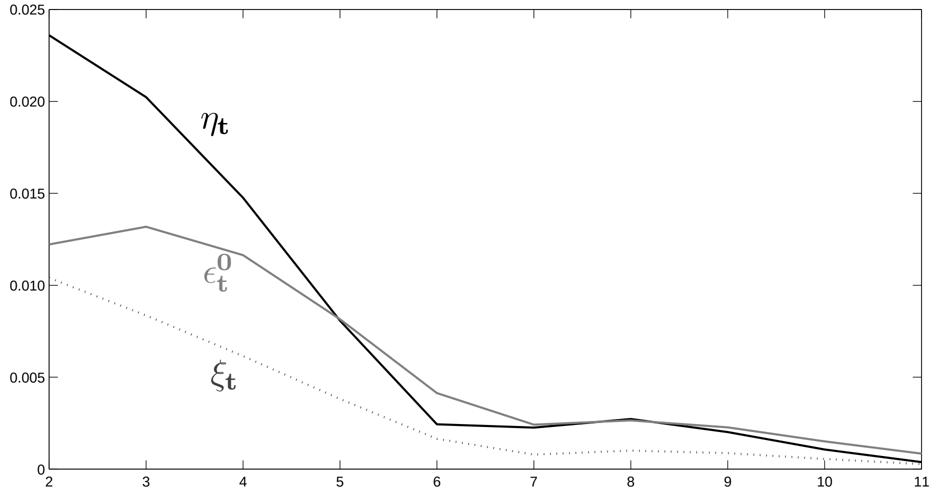


Chart shows the euclidean norm of the change in the impulse response of $p_t^{\bar{k}}$ from a shock to η_t, ϵ_t^0 and ξ_t, respectively, over $T = 20$ periods depending on $\bar{k}$. The values for $\bar{k} = 2$ thus show the norm of the change in the impulse response from p_t^1 to p_t^2 for the three different aggregate shocks.

by first order expectations, thereby not adding much to the dynamics of the price. Similarly, for very imprecise signals, higher-order expectations do not respond significantly to changes in signal values, implying that higher-order thinking does not contribute meaningfully to movements in the endogenous variable. Only for intermediate values of signal precision do higher-order expectations matter greatly for the price dynamics.