Clarification - Quiz #5, Q6

In question 6 of today's quiz you were ask to construct a Confidence Interval for an estimator. Clearly, I left most of us confused by talking about complicated stuff without going over the basic idea first. I am sorry about that and hope the following clarifies things a little.

Introduction

In most of the exercises we covered in the last days, we were constructing some estimator of some parameter and then derived the distribution of this estimator. The reason why we derived the distribution of our estimator was, intuitively, to be able to judge how precise our estimator is and hence have an indication how much we can rely on our estimate.

So now is the first time we try to give answers to those questions. Unfortunately, in this example, we do not know the distribution of our estimator $\bar{X}$. Well, we know that it is $N(\theta, \frac{\sigma^2}{n})$, but this is not helpful, cause we do not know the true σ.

However, instead it can be shown that, whatever the true parameter θ is, the below statistic follows a t-distribution, so[1]

$$\text{T-Stat}_{n-1} = \frac{\bar{X} - \theta}{S/\sqrt{n}} \sim t_{n-1}$$

We can use this knowledge of the above distribution to find, as we did in the last classes, the probability that T-Stat_{n-1} is in some interval $[-b, b]$.

Hyphothesis Testing

Alternatively, we can proceed the other way around and ask: Which would the bounds $-b$ and b need to be, such that the $\Pr(\text{T-Stat}_{n-1} \in [-b, b])$ is e.g. 95%? We call the answer to this question $t^*_{97.5\%,n-1}$ and $t^*_{2.5\%,n-1}$. Since the t-distribution is symmetric around 0, $t^*_{97.5\%,n-1}$ is the same as $-t^*_{2.5\%,n-1}$. To sum up what we have done so far, we have found the critical values t^* such that

$$Pr(-t^*_{2.5\%,n-1} \leq \frac{\bar{X} - \theta}{S/\sqrt{n}} \leq t^*_{2.5\%,n-1}) = 95\%$$

So far, this is rather uninteresting. First of all, who cares about the probability of this complicated expression ending up between some values being 95%, and second of all, we cannot even calculate T-Stat_{n-1} cause we obviously do not know θ (that is what we try to learn about).

But what we can do is to rewrite this to

$$Pr(\theta - C^* \leq \bar{X} \leq \theta + C^*) = 95\%$$

where $C^* = t^*_{2.5\%,n-1} \cdot S/\sqrt{n}$. This is already more useful! Obviously we still do not know θ, but at least it tells us something interesting *if we assume some* θ. Remember that our estimator $\bar{X}$ has a distribution - sometimes it is close to the true θ and sometimes not. The above says, *given some* θ, we can can calculate a left- and right-bound such that the probability that our estimator $\bar{X}$ turns out to be between those bounds is 95%.

Why is this interesting? Because we can come up with the following idea, the idea of **Hypothesis Testing**: Suppose we believe/assume/hypothesize that the true θ is 2 ($H_0 : \theta = 2$). Then we calculate our $\bar{X}$, which is our estimator of θ, and find an estimate of 7. Note, that we might well get this estimate, even if the true θ was 2. However, what we will ask ourselves is: "If θ is really 2, is it

[1] This is obviously only the case in the example we consider! If we look at other estimators in other problems, their distribution will be different.

not quite unlikely to get an estimate of it so far away from 2?!" The answer to this will depend on how precise your estimator is, and this is information is contained in C^*. Suppose that 7 is outside of $[\theta - C^*, \theta + C^*]$, then the above equation answers the question as follows: "Indeed, if 2 is really the truth the probability of getting an estimate as far or further away from 2 as the one we got is less then 5%. Hence, let's give up our idea/assumption/hypothesis." More technically speaking, your decision rule is to *reject the null-hypothesis* $\theta_0 = 2$ if your estimate falls into the area outside the above interval, which is the so-called *rejection region.* And if $7 \in [\theta - C^*, \theta + C^*]$, the answer would be: "No, if θ is in fact 2, then the probability to get a $\bar{X}$ of 7 or further away is actually not that low, it's bigger than 5%."[2]

p-Value

Secondly, let us look at what is the ***p*-value**. It is very similar to the above. But rather than fixing your idea of "quite unlikely" being 5% or less and then finding the rejection region, what we do is, *again given some hypothesized* θ_0 (usually 0), to calculate the probability that the estimator $\bar{X}$ is as far or further away from θ_0 as the estimate we found. That probability is the p-value. Or, to say the same thing differently, we look for the level of significance for which we would be just about to reject the null. Make sure you understand why, if we with an significance level of e.g. 5% reject the null, then the p-value will be less (or equal) than 5%. And if we do not reject at the 5% level, then the p-value is bigger than 5%.

Confidence Interval

Finally, let us look at a third way to see the same thing again, the **Confidence Interval**. For this consider the above equation and rewrite it as

$$Pr(\bar{X} - C^* \leq \theta \leq \bar{X} + C^*) = 95\%$$

So we now lay the very same interval of width $2 \cdot C^*$ around the hypothesized truth, θ_0, rather than our estimate of it. This interval $[\bar{X} - C^*, \bar{X} + C^*]$ is called the **Confidence Interval**. Please make sure you understand that we reject some null about θ in the Hypothesis Testing procedure if and only if this hypothesized value is not in the Confidence Interval. Why does this happen? Once we check whether $\bar{X}$ is more than C^* away from θ_0 and the other time we check whether θ_0 is more than C^* away from $\bar{X}$. The answer will obviously be the same.

Finally, let us look at the correct interpretation of the above statement. Firstly, what it does not say is "The probability that the true value of θ lies in this interval is 95%." It does not say this because θ is not a random variable and hence statements like "with probability x it's here or there" do not make sense. Secondly, what it does say is "With 95% probability we will construct a Confidence Interval that contains the true θ". Another way to interpret the confidence interval is that it contains all values of θ which, if they figure as H_0, would not be rejected in the Hypothesis Testing procedure. Please make sure you get the idea why this is the case.

Interpreted correctly, I personally don't find the Confidence Interval adds a lot to the other two, but I am looking forward to hear your opinion.

[2] This area is sometimes called the "acceptance region", but it is not a particularily good name. Cause we never "accept" a null, we "fail to reject" it.