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CONSUMPTION-SAVINGS DECISIONS WITH QUASI-GEOMETRIC DISCOUNTING

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1. INTRODUCTION

THE PURPOSE OF THIS PAPER is to study how an infinitely-lived consumer with "quasi-geometric" discounting—thought of as represented by a sequence of "selves" with conflicting preferences—would make consumption and savings decisions. In light of experimental evidence suggesting that individuals do not have geometric discount functions (see, for example, Ainslie (1992) and Kirby and Herrnstein (1995)), it is important to understand how departures from geometric discounting affect an individual's consumption-saving decisions.[2]

We assume that time is discrete and that the consumer cannot commit to future actions. We assume that the consumer is rational in that he is able to forecast correctly his future actions: a solution to the decision problem is required to take the form of a subgame-perfect equilibrium of a game where the players are the consumer and his future selves. We restrict attention to equilibria that are stationary: they are Markov in current wealth; that is, current savings cannot depend either on time or on any other history than that summarized by current wealth.

The consumption-savings problem is of the simplest possible kind: there is no uncertainty, and current resources simply have to be divided into current consumption and savings. The period utility function is strictly concave, and the consumer operates a technology for saving that has (weakly) decreasing returns. A special case is that of an affine production function; this special case can be interpreted as one with a price-taking consumer who has a constant stream of labor income and can save at an exogenous interest rate.

Our main finding is one of indeterminacy of Markov equilibrium savings rules: there is a continuum of such rules. These rules differ both in their stationary points and in their implied dynamics. First, there is a continuum of implied stationary points to which the consumer's asset holdings may converge over time. Second, associated with each stationary point is a continuum of savings rules, implying that there is a continuum of dynamic paths converging to each stationary point. We construct these equilibria explicitly—the savings rules are step functions. The discontinuities in the step functions are key: payoff functions with jumps can be optimal precisely because the different selves have conflicting

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[2] The idea that consumers do not discount the future geometrically, and, more generally, exhibit "time inconsistencies," originates with Strotz (1956). Pollak (1968), Phelps and Pollak (1968), Laibson (1994, 1997), and others have elaborated on this idea.

preferences, and depending on how the jumps are structured they can make one self behave more in the interest of another self or vice versa.

The indeterminacy in Markov strategies that we document in this paper is a new finding in the literature on consumption-savings decisions with quasi-geometric discounting.[3] There is also a related literature on differential Markov games, e.g., in applications to models with imperfect altruism (see, e.g., Leininger (1986) and Bernheim and Ray (1987)) and resource extraction (see, e.g., Fudenberg and Tirole (1991)), where existence problems as well as multiplicity have been noted. That literature, however, did not uncover indeterminacy results. Moreover, our equilibrium construction here—whose key elements are discontinuities in decision rules and conflicting objective functions of the players at different points in time—actually suggests that indeterminacy of Markov equilibria can be present in a much larger class of problems than those we study here, including, e.g., optimal fiscal and monetary policy problems where the policy makers at different points in time have conflicting objectives and cannot commit to future policies.

We describe our model in Section 2. In Section 3 we provide the key elements of our equilibrium construction and state a formal theorem summarizing our indeterminacy result. Section 4 provides intuition for our result and Section 5 provides the proof of the theorem.

2. THE MODEL

Time is discrete and infinite and begins at time 0. There is no uncertainty. An infinitely-lived consumer derives utility from a stream of consumption at different dates. We assume that the consumer's self at time t ranks consumption sequences according to

$$u_t + \beta\left(\delta u_{t+1} + \delta^2 u_{t+2} + \delta^3 u_{t+3} + \cdots\right),$$

where the variable u_t denotes the number of utils at time t; these utils are derived from a felicity function $u(c_t)$, where c_t is consumption at time t. In other words, discounting is quasi-geometric: it is geometric across all dates except the current date. We assume that $\delta < 1$ and $\beta < 1$.[4] This formulation embodies an assumption of stationarity: discounting at any point in time has the form $1, \beta\delta, \beta\delta^2, \ldots$. It follows that there is a conflict between selves whenever $\beta \neq 1$; e.g., self t compares consumption at $t+1$ and $t+2$ differently than does self $t+1$.

We assume that the period utility function $u(c)$ is strictly increasing, strictly concave, and continuously differentiable. The consumer's resource constraint is: $c + k' = f(k)$, where k is current and k' next period's capital holdings, f is a strictly increasing, (weakly) concave, and continuously differentiable production function.

[3] Laibson (1994) and Bernheim, Ray, and Yeltekin (1999) find indeterminacy using history-dependent ("trigger") strategies. Harris and Laibson (2000a) provide an existence proof for a consumption-savings problem with uncertainty that does not nest ours. They also show uniqueness when the departure from geometric discounting is sufficiently small. Harris and Laibson (2000b) provide a uniqueness result for a deterministic continuous-time consumption-savings model with "instantaneous gratification" (i.e., the consumer's departure from geometric discounting occurs only in the instantaneous present). Discussions of existence and uniqueness can also be found in Morris (2000) and in Morris and Postlewaite (1997).

[4] Our theorem could alternatively be stated also to include $\beta > 1$, since indeterminacy applies in that case too; see footnote 5 below.

We use recursive methods to study Markov equilibria of the dynamic game between the consumer's different selves. This means that the consumer perceives future savings decisions to be given by a function $g(k)$: $k_{t+l} = g(k_t)$. The Markov assumption is reflected in g being time-independent and having current capital as its only argument. The current self thus solves the "first-stage" problem

$$\max_{k'} u(f(k) - k') + \beta\delta V(k'), \tag{1}$$

where V is an "indirect" utility function: it must satisfy the "second-stage" functional equation

$$V(k) = u(f(k) - g(k)) + \delta V(g(k)). \tag{2}$$

This equation incorporates the assumption that each of the consumer's future selves uses the same savings decision rule g. A Markov equilibrium obtains if $g(k)$ also solves the consumer's first-stage problem for all k.

3. THE STEP-FUNCTION EQUILIBRIUM

In this section, we describe the key elements of our equilibrium construction and state a formal theorem summarizing our indeterminacy result. We also discuss the implications of our indeterminacy result for equilibrium savings paths. Section 4 provides intuition for our result and Section 5 provides a proof of the theorem.

To construct a decision rule, select an arbitrary stationary point for the individual's capital holdings, $\bar{k}$ (we will see that, for some production functions, one needs to restrict the choice of $\bar{k}$ to a bounded range). We will now support $\bar{k}$ as a long-run outcome with a decision rule $g(k)$ which, for $k < \bar{k}$, is a step function with a countable number of steps. Figure 1 illustrates a typical equilibrium step function g.[5,6]

The step function is defined by a countably infinite number of steps at the points $\{s_n\}_{n=0}^{\infty}$. This sequence is strictly increasing and converges to the stationary point $\bar{k}$. Given a sequence $\{s_n\}_{n=0}^{\infty}$, the step function g for savings is thus defined by

$$g(k) = \begin{cases} s_{n+1} & \text{if } k \in [s_n, s_{n+1}) \text{ for } n \in \{0, 1, 2, \dots\}, \\ \bar{k} & \text{if } k \in [\bar{k}, \bar{\bar{k}}], \end{cases} \tag{3}$$

where $\bar{\bar{k}}$ is a constant greater than $\bar{k}$. Given an initial value $k_0 \in [s_m, s_{m+1})$ for some $m \geq 0$ and a decision rule $g(k)$, the equilibrium *time path* of savings levels is then $\{k_0, k_1, k_2, \dots\} = \{k_0, s_{m+1}, s_{m+2}, \dots\}$. In other words, with the possible exception of the initial value k_0, the time path for savings coincides with the step points that define g.

Define the interval

$$I \equiv \left(1 + \frac{1-\delta}{\beta\delta}, \frac{1}{\beta\delta}\right).$$

[5] When $\beta > 1$, the steps are to the right of $\bar{k}$.

[6] The figure depicts g on the interval $[s_0, s_3]$ and on the interval $[\bar{k}, \bar{\bar{k}}]$. The three dots between s_3 and $\bar{k}$ (above the 45° line) indicate that there is an infinite sequence of smaller and smaller steps converging to $\bar{k}$.

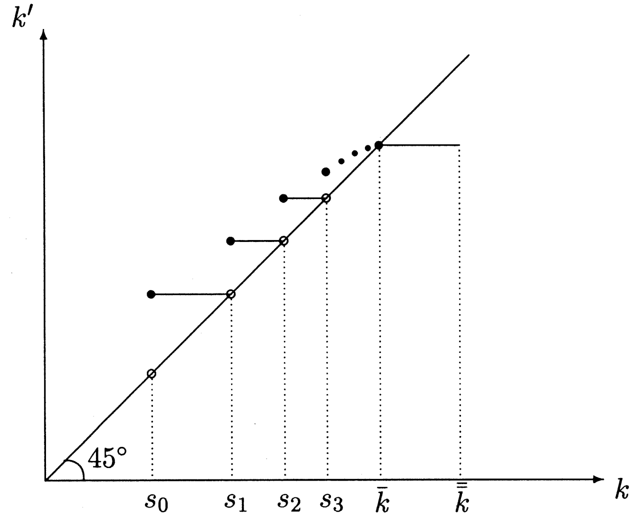

FIGURE 1

I is a nontrivial interval so long as $\beta < 1$ (as β goes to 1, the interval shrinks to one point). The following theorem summarizes our indeterminacy result:

THEOREM: *Consider any $\bar{k}$ such that $f'(\bar{k}) \in I$. Then there exists a left-neighborhood N of $\bar{k}$ such that, for any $x \in N$, there is a decision rule g—defined by a step sequence $\{s_t\}_{t=0}^{\infty}$ according to* (3)*—with $s_0 = x$ that constitutes an equilibrium of the multiple-selves game on the interval $[x, \bar{k}]$. Moreover, different values for x imply different step sequences $\{s_t\}_{t=1}^{\infty}$.*

The theorem is stated in terms of indeterminacy of decision rules g and not in terms of time paths for equilibrium savings, $\{k_t\}_{t=0}^{\infty}$. However, there are three immediate implications for these time paths. First, for any $\bar{k}$ in I, there are initial values k_0 in its neighborhood (selected as x in the theorem) with associated savings paths $\{k_t\}_{t=0}^{\infty}$, each converging to $\bar{k}$. This implication emphasizes indeterminacy of stationary points: many levels of capital can be obtained as long-run outcomes.

Second, for any $k_0 \in I$, there are equilibrium savings paths starting at k_0 and converging to different stationary points. This result follows from the theorem by choosing different stationary points near k_0 with associated overlapping neighborhoods. It emphasizes indeterminacy of equilibrium stationary points from a given initial capital stock.

Third, for any stationary point $\bar{k}$ and any initial k_0 in its neighborhood, there is a continuum of paths starting at k_0 and converging to $\bar{k}$. This property follows from the fact that x in the theorem can be chosen freely and that the step functions resulting from different values for x are distinct (the last statement in the theorem). It emphasizes indeterminacy in dynamics: it shows that a given stationary point can be reached in many ways from a given starting value.

To construct an equilibrium sequence of step points $\{s_n\}_{n=0}^{\infty}$ given an initial step point s_0 in a left-neighborhood of $\bar{k}$, begin by specifying a sequence $\{v_n\}_{n=0}^{\infty}$ of values on the steps: $v_n = V(k_n)$; recall that $\beta\delta V(k')$ is the indirect utility, appropriate for the current self, of saving k'. By definition, thus, the sequence $\{v_n\}_{n=0}^{\infty}$ satisfies

(4) $$v_n = u(f(s_n) - s_{n+1}) + \delta v_{n+1} \qquad (n = 0, 1, 2, \dots).$$

The complete specification of V on the domain $[s_0, \bar{\bar{k}}]$ therefore is as follows:

$$(5) \qquad V(k) = \begin{cases} u(f(k) - s_{n+1}) + \delta v_{n+1} & \text{if } k \in [s_n, s_{n+1}) \\ & \quad \text{for some } n \in \{0, 1, 2, \dots\}, \\ u(f(k) - \bar{k}) + \dfrac{\delta u(f(\bar{k}) - \bar{k})}{1-\delta} & \text{if } k \in [\bar{k}, \bar{\bar{k}}]. \end{cases}$$

Now the step-function construction requires indifference on the steps: $\{s_n\}_{n=0}^{\infty}$ needs to satisfy

$$(6) \qquad u(f(s_n) - s_n) + \beta\delta v_n = u(f(s_n) - s_{n+1}) + \beta\delta v_{n+1}$$

for $n = 0, 1, 2, \dots$. Thus, at s_n the consumer saves s_{n+1} for next period, but is indifferent between this choice and saving s_n.

The restrictions on $\{s_n\}_{n=0}^{\infty}$ can thus be stated as follows: it needs to be strictly increasing and to converge to $\bar{k}$ and there has to exist a sequence $\{v_n\}_{n=0}^{\infty}$ such that the two sequences satisfy the dynamic system given by equations (4) and (6). The indifference requirement formally amounts to a second-order difference equation in the step sequence $\{s_n\}_{n=0}^{\infty}$. The construction of the step sequence thus involves solving this difference equation given an initial condition s_0 and a requirement of convergence to $\bar{k}$. We display this difference equation and discuss it in more detail in Section 4.

4. INTUITION AND DISCUSSION

To show that g prescribes optimal behavior given the perception that g is used in the future, one needs to check all possible one-period deviations. The key elements in the argument, however, can be illustrated with one example: we will argue why it is locally optimal at s_n to choose s_{n+1}, i.e., why self n, if given s_n, will leave self $n+1$ with s_{n+1}, as opposed to slightly higher or lower saving.

First, consider the possibility of saving slightly more, $s_{n+1} + \epsilon$, where ϵ is small and positive. According to the perceived step-function behavior of future selves, this deviation increases the consumption of self $n+1$ by the entire amount ϵ including its return, while at the same time leaving the future capital stocks unaltered. Thus, by this alternative choice, self n would lower his consumption, increase the consumption of self $n+1$, and leave the consumption of all subsequent selves unaltered (consult Figure 1 again to see this). In terms of future capital stocks, we are thus comparing the equilibrium sequence $(s_n, s_{n+1}, s_{n+2}, s_{n+3}, \dots)$ to the deviation $(s_n, s_{n+1} + \epsilon, s_{n+2}, s_{n+3}, \dots)$. Because we are in the neighborhood of a stationary point, consumption in the current and in the next period (along with their associated period marginal utilities) are the same, so it suffices to ensure that the extra ϵ saved gives rise to less than $\beta\delta f'\epsilon$ units of income next period. This is true if $\beta\delta f'(\bar{k}) < 1$. This condition is met by our above restrictions and is central to the proof of Lemma 1 in Section 5.

Second, consider the possibility of slightly lower saving. We will see here how time-inconsistency is a necessary ingredient in supporting the step-function equilibrium. So suppose instead that ϵ is arbitrarily small but negative. This deviation causes next period's saving to fall discontinuously from s_{n+1} to s_n (again, imagine the deviation in Figure 1). Thus, all future consumption levels are affected, since we fall "one step behind" relative to when $\epsilon = 0$: we are now comparing the equilibrium sequence $(s_n, s_{n+1}, s_{n+2}, s_{n+3}, \dots)$ to the deviation $(s_n, s_{n+1} - \epsilon, s_{n+1}, s_{n+2}, \dots)$. Since current

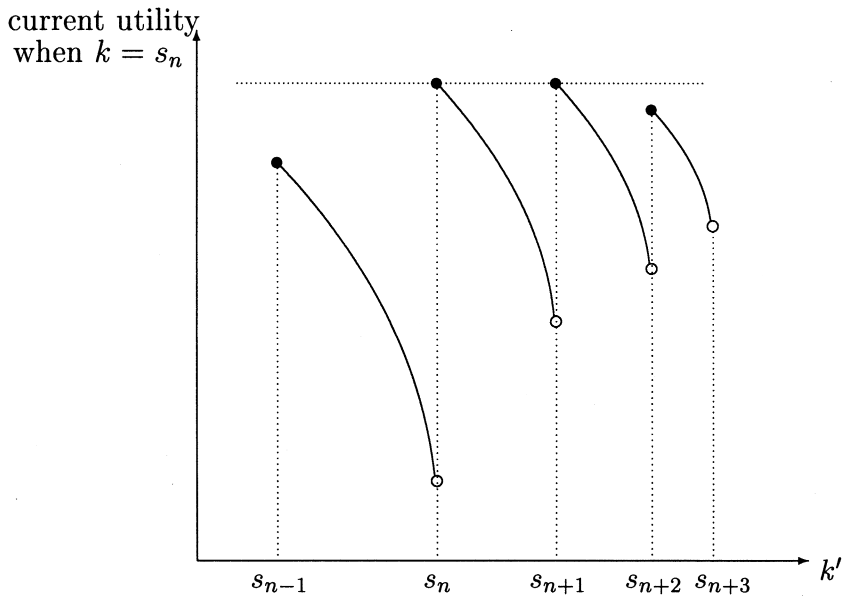

FIGURE 2

consumption is (almost) unaffected, we must decide whether a discontinuous increase in consumption in the next period (and a resulting fall in consumption in the periods after that) is a good deviation. It is not: on the equilibrium path, we have ensured that next period's self is indifferent between remaining on the current step and jumping up one step in saving. The current period's self, however, disagrees with this. He values saving strictly more, so long as $\beta < 1$, since his discount rate between next period and the period after that is δ, not $\beta\delta$, which is the rate used by his next self. Thus, stepping down in saving next period is a move in the *wrong* direction. This kind of argument, together with concavity of the felicity function u, is central to the proofs of Lemmas 2, 3, and 4 in Section 5.

The previous arguments are also illustrated in Figure 2 where we plot the utility, as a function of the chosen savings k', for the current self who is endowed with $k = s_n$. The features that are worth noting are: (i) the individual always chooses a "corner": choosing the left end point of any flat section always dominates choosing an interior point, because $\beta\delta f'(\bar{k}) < 1$, and moving further left would cause a discontinuous drop in utility; (ii) s_n gives the same utility as s_{n+1} (by construction); and (iii) all the other step points are dominated (this can be shown using strict concavity of u). Similar figures can be constructed if the individual starts at intermediate values for current capital. For an initial k slightly above s_n, the values in the figure change so that saving s_{n+1} now *strictly* dominates saving s_n (and vice versa for initial values of k slightly below s_n: now s_n is a strictly better choice).

Finally, one can easily extend the step function to the right of $\bar{k}$ with a flat section (thus implying convergence within one period) over a range. The idea here is the same: the individual goes to a corner in the range $[\bar{k}, \bar{\bar{k}}]$, where $\bar{\bar{k}} > \bar{k}$ solves $u'(f(\bar{\bar{k}}) - \bar{k}) = \beta\delta u'(f(\bar{k}) - \bar{k}) f'(\bar{k})$.

The previous arguments are all straightforward to make in formal detail; we save this for the proof. Moreover, the arguments are of a rather general nature; their essential ingredient is how a disagreement between the two consecutive selves is used to support the discontinuous behavioral rule. This is why we suspect that this equilibrium construction can be used in a larger class of models.

We turn now to a discussion of the admissible interval of stationary points. The two conditions (4) and (6) underlying our step-function equilibrium imply that the sequence of step points must satisfy the following second-order difference equation:

$$\begin{aligned}&\frac{u(f(s_n)-s_n)-u(f(s_n)-s_{n+1})}{s_{n+1}-s_n}\\&=\beta\delta\left\{\frac{u(f(s_{n+1})-s_{n+1})-u(f(s_n)-s_{n+1})}{f(s_{n+1})-f(s_n)}\left(\frac{f(s_{n+1})-f(s_n)}{s_{n+1}-s_n}\right)\right.\\&\qquad\left.+\left(\frac{1}{\beta}-1\right)\frac{u(f(s_{n+1})-s_{n+1})-u(f(s_{n+1})-s_{n+2})}{s_{n+2}-s_{n+1}}\left(\frac{s_{n+2}-s_{n+1}}{s_{n+1}-s_n}\right)\right\}.\end{aligned}$$

This difference equation is similar in structure to the "generalized Euler equation" that Krusell, Kuruşçu, and Smith (2001) use to study differentiable solutions to the multiple-selves game. Defining $c_n \equiv f(s_n)-s_{n+1}$, it says approximately that

$$u'(c_n)=\beta\delta\left\{u'(c_n)f'(s_n)+\left(\frac{1}{\beta}-1\right)u'(c_{n+1})\lambda_n\right\},$$

where $\lambda_n \equiv (s_{n+2}-s_{n+1})/(s_{n+1}-s_n)$ can be viewed as the "marginal" propensity to save along an equilibrium savings path (recall that, with the possible exception of the first period, an equilibrium savings path consists of a subsequence of step points). In other words, the current disutility of saving more has to equal the discounted gain from increased production next period (the first term on the right-hand side) and increased saving next period (the second term on the right-hand side). As n gets large, s_n approaches the stationary point $\bar{k}$: consequently, the three occurrences of u' in this equation cancel in the limit. For the local dynamics of the system (4) and (6) to be monotone and stable, we require that the marginal propensity to save at $\bar{k}$ (i.e., the limit of λ_n) lie in the interval (0,1). It then follows from this equation that $f'(\bar{k})$ has to lie somewhere in the range

$$\left(\frac{1}{\beta\delta}-\frac{1-\beta}{\beta},\frac{1}{\beta\delta}\right).$$

This is exactly the range for which we can construct step-function equilibria.

If f is affine, we are in fact considering a consumer who can save at a gross rate R and has exogenous (perhaps labor) income w. In that case, our theorem applies so long as $R \in I$. If it is, then any value for capital is a stationary point. Thus, the indeterminacy in steady states that obtains in the standard model when the interest rate exactly equals the discount rate obtains here for a range of interest rates.

In a representative-agent, general-equilibrium economy, the condition $f'(\bar{k}) \in I$ specifies a range of steady-state capital stocks. Suppose that capital is in this range, thus delivering an interest rate R in this range. Then it is optimal for the agent to choose any capital stock, in particular the selected one, so we indeed have a steady-state general equilibrium.

5. PROOF OF THE THEOREM

To construct the sequences $\{s_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$, let $\bar{v}$ be the stationary point implied by $\bar{k}$ being the stationary point of the dynamic system given by equations (4) and (6):

$$\bar{v} = \frac{u(f(\bar{k}) - \bar{k})}{1-\delta}.$$

The dynamic system (4) and (6) has to involve convergence to $(\bar{k}, \bar{v})$, and the convergence for capital has to be monotone increasing. Equations (4) and (6) define an implicit function that maps (s_n, v_n) into (s_{n+1}, v_{n+1}) whose Jacobian matrix of first derivatives, evaluated at a stationary point $(\bar{k}, \bar{v})$, has one eigenvalue equal to 1 and one eigenvalue equal to

$$\frac{1-\beta\delta f'(\bar{k})}{\delta(1-\beta)}.$$

This eigenvalue is between 0 and 1 provided that

$$1 + \frac{1-\delta}{\beta\delta} < f'(\bar{k}) < \frac{1}{\beta\delta}.$$

This condition is satisfied by the assumption in the theorem. Under this condition, it is straightforward to modify standard results concerning the local stability of nonlinear difference equations (see, e.g., Scheinkman (1973)) to show that the dynamic system given by (4) and (6) has a one-dimensional stable manifold characterized by a continuously differentiable function $\varphi(s_0, v_0)$. In other words, given a stationary point $(\bar{k}, \bar{v})$, there exists a neighborhood N of this point such that the dynamic system (4) and (6) converges, monotonically, to the stationary point for any initial value $(s_0, v_0) \in N$ satisfying $\varphi(s_0, v_0) = 0$. We can, therefore, construct a continuum of sequences $\{(s_n, v_n)\}$ that converge to a given stationary point. Each of these sequences determines a decision rule and a value function that satisfy the equilibrium conditions in Section 2.

As described in Section 3, given the decision rule g defined by equation (3), the value function V defined by equation (5) satisfies the second-stage functional equation (2) by construction. To check that g solves the first-stage problem (1) given V, four lemmas will be stated and proved. Each lemma considers a specific kind of deviation from the proposed decision rule.

LEMMA 1: *For any $k \in [s_0, \bar{\bar{k}}]$, any choice $k' \in (s_n, s_{n+1})$, for some n, or $k' \in (\bar{k}, \bar{\bar{k}})$, can be improved upon.*

PROOF: Given that $\beta\delta f'(\bar{k}) < 1$, it is always strictly better to be at the left endpoint of an interval than in the interior. For $k \in [\bar{k}, \bar{\bar{k}}]$, the proof of this statement is immediate. For $k \in [s_0, \bar{k})$, we use the fact that $\beta\delta f'(k) < 1$ when k is sufficiently close to $\bar{k}$. *Q.E.D.*

LEMMA 2: *For $k \in [\bar{k}, \bar{\bar{k}}]$, $k' = \bar{k}$ dominates $k' = s_n$ for all n.*

PROOF: We need to prove that $u(f(k) - \bar{k}) - u(f(k) - s_n) \geq \beta\delta(v_n - \bar{v})$. The left-hand side of this expression can be written

$$\sum_{s=0}^{N}\left[u(f(k) - s_{n+s+1}) - u(f(k) - s_{n+s})\right] + u(f(k) - \bar{k}) - u(f(k) - s_{n+N+1})$$

which, since the last two terms cancel as N goes to ∞, equals

$$\sum_{s=0}^{\infty}\big[u(f(k)-s_{n+s+1})-u(f(k)-s_{n+s})\big].$$

The right-hand side of the expression, in turn, can be rewritten as $\beta\delta\sum_{s=0}^{\infty}[v_{n+s}-v_{n+s+1}]$, since v_n goes to $\bar{v}$ as n goes to ∞. Using indifference on the steps, this expression becomes

$$\sum_{s=0}^{\infty}\big[u(f(s_{n+s})-s_{n+s+1})-u(f(s_{n+s})-s_{n+s})\big].$$

It is now clear that the left-hand side is no less than the right-hand side if

$$\begin{aligned}&u(f(s_{n+s})-s_{n+s})-u(f(s_{n+s})-s_{n+s+1})\\&\qquad\geq u(f(k)-s_{n+s})-u(f(k)-s_{n+s+1}),\end{aligned}$$

for each $s\geq 0$. But from the strict concavity of u these inequalities are all met (strictly), since $k>s_{n+s}$ and $\{s_n\}$ is a strictly increasing sequence. *Q.E.D.*

LEMMA 3: *For all n, for $k\in[s_n,s_{n+1})$, $k'=s_{n+s}$ for any $s>1$ or $s<0$ is dominated by $k'=s_{n+1}$.*

PROOF: We start with $s>1$. We need to show that

$$u(f(k)-s_{n+1})-u(f(k)-s_{n+s})\geq\beta\delta(v_{n+s}-v_{n+1}).$$

The left-hand side of this expression can be written

$$\sum_{v=1}^{s-1}\big[u(f(k)-s_{n+v})-u(f(k)-s_{n+v+1})\big]$$

and the right-hand side can be written $\beta\delta\sum_{v=1}^{s-1}[v_{n+v+1}-v_{n+v}]$, which from indifference at steps equals

$$\sum_{v=1}^{s-1}\big[u(f(s_{n+v})-s_{n+v})-u(f(s_{n+v})-s_{n+v+1})\big].$$

Due to strict concavity of u and the sequence $\{s_n\}$ being strictly increasing,

$$\begin{aligned}&u(f(k)-s_{n+v})-u(f(k)-s_{n+v+1})\\&\qquad> u(f(s_{n+v})-s_{n+v})-u(f(s_{n+v})-s_{n+v+1})\big]\end{aligned}$$

for each $v>0$. This implies that the left-hand side exceeds the right-hand side. When $s<0$, by indifference at the initial step, it suffices to show that $u(f(k)-s_n)-u(f(k)-s_{n-s})\geq\beta\delta(v_{n-s}-v_n)$. Now form the same type of sums as for the $s>1$ case, but from $\nu=0$ to $s-1$, and proceed with an analogous argument. *Q.E.D.*

LEMMA 4: *For all n, for $k\in[s_n,s_{n+1})$, $k'=\bar{k}$ is dominated by $k'=s_{n+1}$.*

PROOF: Noting that

$$u(f(s_n)-s_{n+1})-u(f(s_n)-\bar{k})=\sum_{s=1}^{\infty}\big[u(f(s_n)-s_{n+s})-u(f(s_n)-s_{n+s+1})\big]$$

and that

$$\begin{aligned}\beta\delta(\bar{v}-v_{n+1})&=\sum_{s=1}^{\infty}(v_{n+s+1}-v_{n+s})\\&=\sum_{s=1}^{\infty}\big[u(f(s_{n+s})-s_{n+s})-u(f(s_n)-s_{n+s+1})\big],\end{aligned}$$

the result again follows, using concavity and $\{s_n\}$ being an increasing sequence: this results in term-by-term domination. *Q.E.D.*

Lemmas 1–4 suffice to support optimality of the constructed function g on all of its domain. *Q.E.D.*

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