Tax Policy
With Quasi-Geometric Discounting

Per Krusell, Burhanettin Kuruşçu, and Anthony A. Smith, Jr.[1]

April 5

ABSTRACT

We study the effects of taxation in a model with a representative agent with time-inconsistent preferences: discounting is quasi-geometric. Utility is derived from consumption and leisure, and taxation can be based on consumption and investment spending as well as on capital and labor income. The model allows for closed-form solutions, and welfare comparisons can be made across different taxation systems.

Optimal taxation analysis in this model leads to time-inconsistency issues for the government, assuming that the government shares the consumer's preferences and cannot commit to future taxes. We study time-consistent policy equilibria for different tax constitutions. A tax constitution specifies what tax instruments are available, and we assume that the government can commit to a tax constitution. The results show that a constitution leaving the government with no ability to tax results in strictly higher welfare than one where the government has full freedom to tax. Indeed, for some parameter values, the best tax constitution of all is laissez-faire (even though the government is benevolent and fully rational). For other parameter values, it may be optimal to allow the government to use a less than fully restricted set of tax bases.

1 Introduction

Some recent literature emphasizes the possibility that individual consumers have time-inconsistent preferences.[2] One aspect of time-inconsistency in preferences is that consumers want to commit their future consumption levels. To the extent that they cannot, but are still rational in trying to resolve their dilemma, an interesting question is whether there is a role for government intervention in markets. In other words, could government policy help improve outcomes for private consumers? Suppose that the government cannot directly make choices for the consumer, but that it can affect the consumer's choices indirectly by using taxes on different activities. Suppose, moreover, that the government cannot commit to specific future tax sequences; if it could, it would, in general, be able to help the consumer

[1]We thank Jang Ok Cho and Tack Yoon for very helpful comments.

[2]The ideas were first brought up in Strotz (1956), then developed in Phelps and Pollak (1968), and, more recently, examined in a sequence of papers by Laibson, starting with Laibson (1994), and in Barro (1997).

fully overcome his private inability to commit. In this situation, would the government be able to help private consumers?

In a previous paper, Krusell, Kuruşçu, and Smith (1999), we showed, using a particular model, that if the government can commit to future taxes, then there exists taxation policy that would make private individuals better off.[3] In the absence such an ability to commit, however, a benevolent, rational government would, left to its own devices, choose a tax policy leaving the consumers *worse off* than without government intervention. In other words, we demonstrated that we have a new case of a nontrivial rules-vs.-discretion situation: a benevolent government cannot avoid selecting certain policies that result in an equilibrium that is worse than without intervention. In other words, a simple rule that would do better than discretion is to have no government at all.

The purpose of the present paper is twofold. First, we extend our setup to a more general economy where labor supply is modeled explicitly and explore whether it is still true that, in our specific sense, a benevolent government with the ability to tax is bad for the economy. Second, we develop our rules-vs.-discretion discussion by comparing different tax constitutions. For example, if we could rule out taxation of capital income, but still allow the government to tax other income sources, would the equilibrium allocation improve?

Our results here show that our earlier finding that no government at all is best (provided the government is benevolent and tries to help out) does not generalize, at least not for all parameter configurations of the model. As in our earlier paper, we can solve the model explicitly and globally. This is considerably more difficult with endogenous labor supply and with more tax rates available at the government's discretion (restricting taxation to be proportional). We do fully generalize the earlier finding that with unrestricted tax bases, that is, with the largest number of instruments available, the outcome resulting from benevolent government action is strictly worse than with no government at all. However, gradual restrictions of tax bases, such as eliminating the possibility of taxing capital income, may or may not improve over no restriction at all, and they may or may not be better than laissez-faire.

The nature of our welfare result has a "second-best" flavor: fewer restrictions are not necessarily worse, because fewer restrictions can give the government distorted incentives, and distorted incentives may lead to better outcomes in a world where the government is tempted to take the wrong kind of action. For example, if the government can only tax capital income and labor income (at different rates), it has no possibility to affect the current savings rate in our economy, which it would like to decrease (assuming that it, like the consumer, has a bias toward the present). It may then instead choose to tax labor, so that agents work less and achieve higher current utility (this occurs when the government, like the consumer, has a bias toward the present). For the budget to balance, this implies that capital income will be subsidized. Foreseeing such government behavior in the future, savings rates will be higher than with no taxation at all, and may therefore give higher welfare.

We also find, for a tax constitution allowing a tax on capital income and a tax on labor

[3] Laibson (1996) studies policy in the context of a model similar to the one considered here, but he restricts his attention to cases where the government can commit to future taxes. He also restricts the technology available to consumers to be linear in savings; we consider the neoclassical growth model with decreasing returns to capital.

income, that there may be more than one equilibrium: in and of themselves, expectations of how future governments will tax will determine how the current government taxes. Multiple equilibria, however, are rare: they only appear in a very small subset of the parameter space we consider.

Section 2 describes the model setup. Section 3 solves for competitive equilibria for a given set of constant taxes. This section is an important building block for the ensuing analysis, since taxes will be constant when they are chosen by the government. Section 4 looks at a planning problem: a situation in which a central planner can directly choose allocations in the current period. The results are useful as a comparison to the case where the planner chooses allocations indirectly—by means of tax rates—within a decentralized mechanism. Section 5 formalizes time-consistent policy equilibria, and solves for such equilibria for different tax constitutions. It compares the time-consistent policy equilibria with outcomes under full and partial commitment, respectively. With partial commitment, we have in mind more specific constitutions: the constitution would not only specify tax bases, but also a set of associated (constant) tax rates that would be committed to. Section 6 concludes.

2 The economy

2.1 Preferences

We consider a discrete-time economy without uncertainty. A representative consumer, who is alive at each date, obtains utility from current and future consumption and leisure. Preferences are time-additive and can be described as follows. At time zero, when the economy starts, the consumer experiences a utility given by

$$U_0 = u_0 + \beta\left(\delta u_1 + \delta^2 u_2 + \delta^3 u_3 + \ldots\right),$$

where u_t is the utility stream from consumption and leisure at time t. Notice that this formulation "almost" has a geometric form—it is "quasi-geometric": the discount weights are geometric, with rate δ, across utils one period from now and on, but the discount rate between the current and next period's utils is not, in general, given by δ, but instead by $\beta\delta$, where β could be larger or less than 1.

Preferences are stationary over time in the sense that one period later, they take exactly the same form:

$$U_1 = u_1 + \beta\left(\delta u_2 + \delta^2 u_3 + \delta^3 u_4 + \ldots\right).$$

Thus, the consumer evaluates utility at period 1 to be U_1: it also places a special weight on the first period, and has geometric discounting thereafter. Similarly,

$$U_2 = u_2 + \beta\left(\delta u_3 + \delta^2 u_4 + \delta^3 u_5 \ldots\right),$$

is perceived by the consumer at time 2, and so on.

When $\beta = 1$, these preferences reduce to the standard geometric preferences typically used in macroeconomic models. When $\beta \neq 1$, however, there is a time-inconsistency in the consumer's preferences. Why? At date 0, the comparison between utils at dates 1 and 2 is not made in the same way as at date 1. Suppose, for example, that we have

$\beta < 1$. Then, the consumer places higher weight on consumption/leisure at date 1 (relative to consumption/leisure at date 2 and later dates) when he is at date 1 than when he was at date 0. That is, the consumer appears prone to immediate gratification: he is particularly impatient in the very short run.

We have considered the simplest possible deviation from pure geometric discounting: consumers at time t and $t+k$ agree on the relative values of goods at all dates, except those involving date $t+k$ itself. Our quasi-geometric class is naturally generalized gradually by changing the last part of the previous statement to "except those involving date $t+k$, $t+k+1$, ..., and $t+k+K$" by introducing further factors β_2 through β_{K+1}, labeling our original β as β_1.

An alternative interpretation of our preferences, which relies explicitly on dynastic arguments and which we may all see as carrying some intrinsic elements of truth, is as follows. Suppose that a dynasty is composed of individuals living only one period, after which they are replaced by their offspring. The utility functions just described might then capture how each individual cares about the consumption and leisure enjoyed by himself and his offspring. The key point here is that he cares differently about the consumption and leisure levels of his children, grandchildren, and so on than what his children and grandchildren do themselves.

The dynastic interpretation does not involve the label time-inconsistent in a direct way, but rather emphasizes disagreement across agents, in this case through "impure" altruism. Given this disagreement, it is natural to think of the interaction between individuals of different generations but within the same dynasty in terms of a dynamic game. This is also how we will model decision making here.

We assume that the period utility function is

$$u(c, 1-l) = \lambda \log(c) + (1-\lambda) \log(1-l),$$

where c is consumption and l is labor effort. The reason for our restriction to logarithmic preferences is that it allows closed-form solutions of all our equilibria. It is possible to generalize to preferences with constant elasticity of intertemporal substitution (assuming that leisure is not valued) and still obtain closed-form solutions, if the production function is linear in capital.

2.2 Technology

We assume that the aggregate resource constraint in the economy is

$$c + k' = Ak^{\alpha} l^{1-\alpha},$$

where primes denote consecutive-period, as opposed to current, values. That is, we assume Cobb-Douglas production, with full depreciation of capital after it has been used in production in the current period. We assume the capital share α to be strictly between zero and one.

2.3 Markets

We assume perfect competition in product markets, which are the only markets in operation. Perfect competition implies marginal-product pricing of capital and labor inputs:

$$r = \alpha A k^{\alpha-1} l^{1-\alpha}$$

$$w = (1-\alpha) A k^{\alpha} l^{-\alpha}.$$

The consumer can buy capital, store it, and realize the returns r' in the next period. That is, capital works as (and is indistinguishable from) a one-period asset for the consumer.

One could introduce multiperiod assets into this economy, but we refrain from explicit modeling of such assets in this paper. Such assets would be priced on the basis of arbitrage, using the returns on any one-period assets. Such arbitrage works if there exist one-period assets in any time period, and if assets can be retraded prior to maturity. Therefore, the introduction of multiperiod assets does not alter equilibrium allocations in this model under these assumptions.

2.4 Government

We assume that there is a government with the ability to tax each source of income and expenditure at proportional rates. That is, we allow consumption taxes, taxes on investment, taxes on capital income, and taxes on labor income. One of these is clearly redundant, as it can be expressed as a combination of the others; consumption taxes are therefore assumed to be zero throughout the paper.

The government has a budget constraint, which is assumed to balance every period. There are no expenditures other than possible subsidies introduced as negative taxation of one of the tax bases. With the obvious notation, the government budget reads

$$rk\tau_k + wl\tau_l + k'\tau_i = 0.$$

2.5 The consumer's budget and his choice variables

The consumer's current budget reads

$$c + k'(1+\tau_i) = rk(1-\tau_k) + wl(1-\tau_l).$$

His decision problem involves how to choose c, k', and l to maximize his utility. We assume that the consumer cannot directly choose future consumption or leisure levels. Indirectly, however, the consumer can affect these variables by his savings decision, to the extent that next period's asset holdings influence subsequent consumption and leisure choices. These assumptions amount to a lack of commitment technology for future choices. The lack of commitment is a friction in this model; with commitment, standard welfare theorems would apply, and there would be no role for government policy.

2.6 Decisions

Having described the physical environment, how are decisions made in this economy? We adopt the following principles. First, the consumer is rational: he foresees that his future selves (or descendants) have preferences that do not agree with his own, and that these consumers will have the power to make future decisions (given the lack of commitment). Second, we model the interaction between the different selves as a dynamic game. We require equilibria of this game to satisfy three properties: (i) that they be subgame perfect; (ii) that they be first-order Markov (with capital, or asset holdings, being the only state variable); and (iii) that they be a limit of the corresponding finite-horizon game.

The first of these requirement simply formalizes the rationality with which we want to endow consumers. We view rationality as hard to dispense with: why would the consumer systematically make mistakes in foreseeing his own future behavior? Alternatives to this approach are represented in the literature, however; see, e.g., O'Donoghue and Rabin (1999).

The second requirement can most simply be seen as a way of reducing the set of equilibria. In the kind of dynamic games we consider, trigger-strategy equilibria are possible; indeed, Bernheim, Ray, and Yeltekin (1999) provide examples of such equilibria in a very similar model. Is it warranted to rule out trigger-strategy equilibria? In our setup, they would take the following form: I am induced to save more today because I expect my future self to punish me by not saving, if I do not save today. In turn, my future self is expected to go through with such punishment since his future selves threaten to otherwise punish him, and so on. One could argue that these strategies have an unusual flavor in this case: why would the consumer feel bound by expectations about his future behavior, when he himself is, or will be, free to alter this behavior? Why not just "restart" the expectations at any point in time, if it is in the consumer's interest? Such an argument would select the best equilibrium, given any initial capital stock, and the remaining state variable could not, then, contain anything but this capital stock. However, allowing such "renegotiation" with yourself explicitly in the equilibrium definition is not trivial, and we do not know whether there exists a definition justifying our elimination of history-dependent equilibria. In addition, one may not even want to use this principle: some very good equilibria—relying on history dependence—may be ruled out.[4]

The third requirement is necessary to prevent the equilibrium set from being too large: dynamic consumption-savings games actually lead to a continuum of first-order Markov solutions (again, with capital as the only state variable). This indeterminacy is discussed in Krusell and Smith (1999). Among all Markov equilibria, we select the unique equilibrium that is the limit of finite-horizon equilibria. Other selection criteria are possible; for instance, one could again use a renegotiation argument and select on the basis of utility, for each level of capital. We do not, however, even though the equilibrium we focus on here is (at least locally) dominated in utility. The reason for our choice is threefold: (i) no other equilibria than the one we study can be solved for analytically; (ii) the other equilibria that we know of have the somewhat unattractive feature that the savings rule is discontinuous; and (iii) it is not clear that equilibria can be meaningfully ranked based on utility: equilibrium A may

[4]As an illustration, the principle we use here would make cooperation impossible in the repeated prisoner's dilemma, since that game has no physical state variable.

be better than equilibrium B as perceived by a consumer with a given capital holding, but this equilibrium may lead the capital stock to take on a future value where the ranking is reversed.

3 Recursive competitive equilibrium with exogenous taxes

We now assume taxes to be exogenously given and set at constant rates satisfying the government budget. It might not be obvious that the government budget can be satisfied at all possible times for constant tax rates (other than the zero rates), but it is true in our environment, as shall become clear shortly. Later in the paper, we will support specific tax rates as time-consistent tax policy equilibria, when taxes are chosen by a government trying to maximize consumer utility.

3.1 Formulating the consumer's problem

We use recursive methods to describe the consumer's decision problem of the consumer. Assuming first that the consumer is in a steady-state environment with constant prices r and w (the steady-state assumption is only used for illustration here and will be dropped momentarily), this decision problem can be characterized by the functions V_0, g_i, g_l, and V. The consumer's problem is to solve

$$\max_{k',l}\{u(rk+wl-k',1-l)+\beta\delta V(k')\}\equiv V_0(k),$$

where

$$V(k)=u(rk+wg_l(k)-g_i(k),1-g_l(k))+\delta V(g_i(k)),$$

given the two functions g_i and g_l. These functions summarize how the consumer perceives the behavior of his future selves. Notice that these functions are time-independent and only depend on capital (the first-order Markov assumption, with capital as the state variable). The function V is the indirect utility function used by the current consumer to evaluate the effects of leaving his next self with different amounts of capital. Given g_i and g_l, V can be solved for from the second equation above (it is a contraction mapping). It can then be substituted into the first expression and the maximization can be executed. Notice that V is computed discounting the future at rate δ—the appropriate rate for the current consumer to use for dates beyond the current date—whereas that entire future is discounted with respect to the present (as expressed in V_0) at rate $\beta\delta$.

When the solutions to the consumer's decision problem above reproduce $g_i(k)$ and $g_l(k)$ for every k, we have a subgame-perfect, first-order Markov equilibrium in the dynamic game between the consumer and his future selves. We will now proceed to define a general equilibrium for this economy, on and off the steady state.

3.2 General equilibrium

We continue to use recursive methods. The state variable of the economy is aggregate capital, $\bar{k}$. The relevant state variable for the consumer in making his decisions is therefore $(k, \bar{k})$, with k representing his own current capital holding.

There are two equilibrium functions describing how aggregate quantities are determined: G_i describes the law of motion of aggregate capital, $\bar{k}' = G_i(\bar{k})$, and G_l describes aggregate labor supply, $\bar{l} = G_l(\bar{k})$. Prices, $r(\bar{k})$ and $w(\bar{k})$, are given by marginal products off the production function, evaluated at equilibrium quantities.

The consumer's decision problem can be written as follows.

$$V_0\left(k, \bar{k}\right) = \max_{k', l}\{\lambda \log\left(r(\bar{k})k\left(1-\tau_k\right) + w(\bar{k})l\left(1-\tau_l\right) - k'\left(1+\tau_i\right)\right) +$$

$$\left(1-\lambda\right)\log\left(1-l\right) + \beta\delta V\left(k', G_i(\bar{k})\right)\}$$

In turn, $V\left(k, \bar{k}\right)$ satisfies

$$V\left(k, \bar{k}\right) = \lambda \log\left(r(\bar{k})k\left(1-\tau_k\right) + w(\bar{k})g_l(k, \bar{k})\left(1-\tau_l\right) - g_i\left(k, \bar{k}\right)\left(1+\tau_i\right)\right) +$$

$$\left(1-\lambda\right)\log\left(1-g_l(k, \bar{k})\right) + \delta V\left(g_i\left(k, \bar{k}\right), G_i(\bar{k})\right).$$

Definition 1 *A recursive competitive equilibrium for this economy consists of a set of decision rules, $g_i\left(k, \bar{k}\right)$ and $g_l\left(k, \bar{k}\right)$, a value function, $V\left(k, \bar{k}\right)$, pricing functions $r(\bar{k})$ and $w(\bar{k})$, a law of motion for aggregate capital, $G_i\left(\bar{k}\right)$, and an aggregate labor supply function $G_l(\bar{k})$ such that*

1. *$V\left(k, \bar{k}\right)$, $g_i\left(k, \bar{k}\right)$, and $g_l\left(k, \bar{k}\right)$ solve the consumer's maximization problem;*
2. *firms maximize, i.e., $r(\bar{k}) = f_1\left(\bar{k}, G_l(\bar{k})\right)$ and $w(\bar{k}) = f_2\left(\bar{k}, G_l(\bar{k})\right)$;*
3. *the consumer's savings decisions are consistent with the law of motion for aggregate capital, i.e., $g_i\left(\bar{k}, \bar{k}\right) = G_i\left(\bar{k}\right)$; and*
4. *the consumer's labor supply decision is consistent with the aggregate labor supply function, i.e., $g_l\left(\bar{k}, \bar{k}\right) = G_l\left(\bar{k}\right)$.*

Given the specific functional forms adopted here, we can solve explicitly for all equilibrium elements. We have

Proposition 1 *For our parametric economy, the recursive competitive equilibrium is given by:*

1. *$V\left(k, \bar{k}\right) = a + b\log\bar{k} + c\log\left(k + \varphi\bar{k}\right)$, where $c = \frac{1}{1-\delta}$, $b = \frac{\alpha\delta+\alpha\lambda-\alpha\delta\lambda-1}{(1-\delta)(1-\alpha\delta)}$, and $\varphi = \frac{(1-\alpha)(1-\delta(1-\beta))(1-\tau_l)}{\alpha(1-\delta)(1-\tau_k)\bar{l}}$ ($\bar{l}$ is defined below);*

2. $g_i\left(k,\bar{k}\right) = \frac{\delta\beta}{1-\delta(1-\beta)}\frac{1-\tau_k}{1+\tau_i} r\left(\bar{k}\right)k$;

3. $G_i\left(\bar{k}\right) = \frac{\alpha\delta\beta}{1-\delta(1-\beta)}\frac{1-\tau_k}{1+\tau_i} A\bar{k}^{\alpha}\bar{l}^{1-\alpha}$;

4. $g_l(k,\bar{k}) = \lambda - \frac{\alpha(1-\lambda)(1-\delta)(1-\tau_k)}{(1-\alpha)(1-\tau_l)(1-\delta(1-\beta))} \cdot \bar{l} \cdot \frac{k}{\bar{k}}$; *and*

5. $G_l(\bar{k}) = \bar{l} \equiv \frac{\lambda(1-\alpha)(1-\delta+\delta\beta)(1-\tau_l)}{(1-\alpha)(1-\delta+\delta\beta)(1-\tau_l)+\alpha(1-\lambda)(1-\delta)(1-\tau_k)}$.

Proof: See the appendix.

The equilibrium has several noteworthy properties. First, the savings rate out of total production in this economy is constant. This is an important property, indeed a necessary ingredient, for studying the problem analytically; for instance, the time-consistent policy equilibrium below could not be studied otherwise. The competitive equilibrium savings rate, out of total output, is given by $\frac{\alpha\delta\beta}{1-\delta(1-\beta)}\frac{1-\tau_k}{1+\tau_i}$; the savings rate out of net-of-tax capital income in competitive equilibrium (CE), which we denote s^{CE}, equals $\frac{\delta\beta}{1-\delta(1-\beta)}$.

Second, and relatedly, individuals' savings behavior, given aggregate paths, are determined as a linear function of their own capital holdings: they are a constant fraction of the individual's current capital income. The aggregate capital stock has a negative effect on individual savings, by the lowering of the return to capital. The simple expression for individual savings—in particular, that wage rates or wage income do not appear—is a feature of Cobb-Douglas production and of logarithmic preferences: it is straightforward to check that with consumers consuming a constant fraction of their total net-present income (including wages, current and future) in equilibrium, the savings rate out of current capital income alone actually becomes constant, as future wage and interest rates reduce to simple forms and end up canceling. The aggregate capital stock also does increase current wages, and, in this sense should have a second avenue for affecting savings. However, it also affects future wages, via the induced increases in future capital stocks, as well as the interest rates used to discount these increases to the present. All these increases in wage income make the individual better off, but given Cobb-Douglas production, increases now and in the future all lead in the same direction and occur in such a way that a consumer with logarithmic preferences chooses to not let them affect savings.[5] What remains is just the instantaneous depressing effect on capital income.

Third, labor supply is also determined in a very convenient way. In equilibrium, total labor supply is constant, that is, it is independent of the capital stock. This is also a result of the preference assumptions; income and substitution effects cancel in equilibrium. Higher capital increases the wage rate, thus making the agent substitute toward less leisure and more work; however, it also has a wealth effect leading to more leisure and a lower labor supply. These effects can be seen more clearly in the expression for $g_l(k,\bar{k})$. We see that k has a negative effect only: it is a pure wealth effect (no prices are affected). We also see that $\bar{k}$ has a positive effect, as it leads to substitution away from leisure; $\bar{k}$ also has a wealth

[5]With other kinds of preferences, the increase in the rate of return on savings would influence savings; with logarithmic preferences, income and substitution effects cancel, and rates of return do not affect savings.

effect, but it is dominated by the substitution effect. In equilibrium $k = \bar{k}$ and the two effects can be seen to cancel exactly in our expression.

It is also useful for the analysis below to display how the labor decision interrelates with the equilibrium savings rate (out of net capital income): $\bar{l}$ is derived to satisfy

$$\bar{l} = \frac{\lambda (1-\alpha)(1-\tau_l)}{(1-\alpha)(1-\tau_l) + \alpha (1-\lambda)(1-s^{CE})(1-\tau_k)},$$

and the expression for $\bar{l}$ in the proposition above is obtained by substituting in for s^{CE}. For example, this shows that if the preference parameters were altered so as to make the savings rate increase in equilibrium, equilibrium labor supply would also increase.

Fourth, what are the effects of individual tax rates in this economy? We see that the tax rate on capital reduces savings. This occurs because savings out of capital income are constant, as pointed out above, and the current tax on capital lowers capital income. This is a lump-sum effect: the only reason why savings and consumption both go down is that the tax reduces wealth, ceteris paribus. The investment tax does not change the total amount of consumption foregone, but it directly lowers the relative price of obtaining capital tomorrow, and thus less such capital is obtained. Labor taxes do not influence savings directly. Again, as explained above, this is due to the fact that proportional changes in wages, today and in the future, leave a consumer with logarithmic preferences with the same savings.

Of course, taxes interact via the government budget constraint, and the equilibrium effect of changing a given tax is more complex. We will return to this issue below.

4 Laissez faire vs. planning outcomes

We now compare the welfare properties of the competitive equilibrium without government intervention, that is, with all taxes set to zero, to an outcome a central planner would dictate. A central planner here is interpreted as a representative of the consumer who has the power to dictate current savings and labor supply decisions. Like the consumer himself, the planner cannot directly affect future savings and labor supply decisions; he can only do this indirectly, by affecting capital accumulation. Thus, we assume that the central planner is subject to the same lack of commitment as the consumer, and that he rationally takes his future savings and labor supply decision rules as given and generated by rational behavior of his future selves.

4.1 The central planner's allocation

The recursive formulation of the central planning problem consists of four functions: V_0, V, h_i, and h_l. These parallel the corresponding functions for the competitive consumer (we use the same notation for the value functions, denoting competitive consumers' decision rules with g's and planners' decision rules with h's). The planner's problem is written

$$V_0(k) \equiv \max_{k',l} \{\lambda \log(f(k,l) - k') + (1-\lambda) \log(1-l) + \beta\delta V(k')\},$$

where

$$V(k) = \lambda \log(f(k, h_l(k)) - h_i(k)) + (1-\lambda)\log(1 - h_l(k)) + \delta V(h_i(k)).$$

The fixed point condition here, as in the competitive consumer's case, is that the values for k' and l appearing from this maximization problem equal $h_i(k)$ and $h_l(k)$, respectively: perceptions of future behavior are consistent with actual behavior.

It is straightforward to show that the following functions are solutions to the planning problem:

$$k' = h_i(k) = \frac{\beta\delta}{1 - \alpha\delta(1-\beta)} \alpha A k^{\alpha} l^{1-\alpha}$$

and

$$V(k) = a + b \log k.$$

The central planning (CP) savings rate, out of net capital income, is given by

$$s^{CP} = \frac{\delta\beta}{1 - \alpha\delta\,(1-\beta)}.$$

That is, it is lower than the competitive equilibrium savings rate if β is less than one, and higher if β equals one; if β equals one, the planner saves as much as do competitive consumers. We will return to an interpretation of this finding below. Notice that the result is only obtained if α is strictly smaller than 1.

The labor decision made by the planner is as follows:

$$l^{CP} = \frac{\lambda\,(1-\alpha)}{1 - \alpha + \alpha\,(1-\lambda)\,(1 - s^{CP})} = \frac{\lambda\,(1-\alpha)\,(1 - \alpha\delta\,(1-\beta))}{(1-\alpha\lambda)\,(1-\alpha\delta) + (1-\alpha)\,\alpha\delta\beta\lambda}.$$

Note here that the relation between the labor choice and the savings rate choice is identical to that for the competitive consumer. This means, in particular, that if savings rates are the same in the two cases, so are labor choices. Furthermore, the competitive equilibrium labor supply is higher (lower) than the central planner's labor choice whenever β is less (larger) than one.

We summarize these findings and compare the utility levels across the two allocations in the following proposition:

Proposition 2 *We find that:*

1. *savings rates differ as follows:* $s^{CE} > s^{CP}$ $\left(s^{CE} < s^{CP}\right)$*; and that labor choices differ as follows:* $\bar{l}^{CE} > \bar{l}^{CP}$ $\left(\bar{l}^{CE} < \bar{l}^{CP}\right)$*, for* $\beta < 1$ $(\beta > 1)$*;*

2. *and for all current values of the capital stock, and whenever* $\beta \neq 1$*, the competitive outcome results in strictly higher welfare for the consumer than does the central planning outcome.*

Proof: See the appendix.

The welfare comparison in the proposition is made in terms of the utility of the consumer making the current decisions. One might also wonder how this consumer's future selves feel about the comparison. If β is less than 1, then both current and all future consumers prefer the competitive equilibrium outcome. The reason is that in this case, savings are higher in the competitive equilibrium, which is an additional reason for future generations to prefer competition: for any given capital stock, they prefer competition, and with more capital under competition, the utility gap increases. If β exceeds 1, then the utility gap is decreased, and the consumer's future selves might prefer the central planning outcome.

4.2 Understanding the results: the generalized Euler equation

To analyze the determinants of the result that a central planner, despite his direct command of resources, and despite his benevolence, would harm rather than help the consumer, we will characterize the decision making with Euler equations. These Euler equations are "generalized" here; they do not look quite like the Euler equations we are familiar with from the $\beta = 1$ case.[6]

To derive these Euler equations for the planner, first note that the first-order condition for savings, evaluated at the optimal choices, reads

$$u_1(f(k, h_l(k)) - h_i(k), 1 - h_l(k)) = \beta\delta V'(h_i(k)).$$

The first-order condition for labor reads

$$f_2\,(k, h_l(k))\, u_1(f(k, h_l(k)) - h_i(k), 1 - h_l(k)) = u_2(f(k, h_l(k)) - h_i(k), 1 - h_l(k)).$$

Given the expression for $V(k)$ in the planner's problem, which holds for all k, we can obtain an expression for $V'(k)$:

$$V'(k) = u_1(f(k, h_l(k)) - h_i(k), 1 - h_l(k))(f_1(k, h_l(k)) + f_2(k, h_l(k))h_l'(k) - h_i'(k))$$
$$-u_2(f(k, h_l(k)) - h_i(k), 1 - h_l(k))h_l'(k) + \delta V'(h_i(k))h_i'(k).$$

This expression normally simplifies a great deal, due to the envelope theorem: the indirect effects, that is, those terms involving $h_i'(k)$ and $h_l'(k)$ (and therefore $V'(k')$), would cancel. They would cancel because the future values of k' and l would normally be chosen so as to maximize current utility. This is not necessarily the case here, however: they are chosen by a different self, one with different preferences.

How do preferences between current and future selves differ? They differ in the intertemporal aspects; hence, the indirect effects of altering the intertemporal decisions of the future self cannot be ignored here. However, they do agree on the labor choice: the consumption-leisure decision is a static decision. Formally, we use the first-order condition for labor to eliminate all terms involving $h_l'(k)$.[7]

[6] David Laibson was the first to use this terminology and approach.

[7] In a model with human capital accumulation, these effects would not cancel if the work decision in that case had intertemporal effects (say, by lowering the amount of time spent on education, or by raising the amount learnt on the job). Also, in a model where parents have different consumption-leisure preferences for their children than their children have for themselves (parents might not appreciate the leisure to the same extent), we would also have additional effects.

Next, substitute the condition for $V'(h_i(k))$ from the first-order condition for savings into the equation just obtained, so that $V'(k)$ can be solved for. Then, update one period forward and substitute back into the first-order condition for savings. This leads to the generalized Euler equation:

$$
\begin{gathered}
u_1(f(k,h_l(k)) - h(k), 1 - h_l(k)) = \\
\delta u_1(f(h_i(k), h_l(h_i(k))) - h_i(h_i(k)), 1 - h_l(h_i(k))) \cdot \\
[\beta f_1(h_i(k), h_l(h_i(k))) + (1-\beta) h_i'(h_i(k))].
\end{gathered}
$$

This equation, which is written as a functional equation in the decision rules h_i and h_l—it holds for all k—can be more compactly stated as

$$u_1(c, 1-l) = \delta u_1(c', 1-l')(\beta f_1(k', l') + (1-\beta) h_i'(k')).$$

That is, we have the standard Euler equation for $\beta = 1$, but whenever β differs from 1, there is an occurrence of h_i': it matters to the current consumer how he can affect the savings behavior of his future self. For instance, with $\beta < 1$, he thinks his future self will not save enough. Therefore, there is an added benefit for him from saving today: he increases future savings (since h_i is always increasing). This is the fundamental new ingredient in the savings decision of consumers with time-inconsistent preferences who rationally foresee their future decisions.[8]

For the competitive agent, the first-order condition for leisure of the competitive consumer has the same form as the condition for the planner:

$$f_2(k, G_l(k))\, u_1(f(k, G_l(k)) - G_i(k), 1 - G_l(k)) = u_2(f(k, G_l(k)) - G_i(k), 1 - G_l(k)).$$

A competitive-equilibrium generalized Euler equation can be derived using the same logical steps as for the planner. We obtain

$$
\begin{gathered}
u_1(f(k, G_l(k)) - G_i(k), 1 - G_l(k)) = \\
\delta u_1(f(G_i(k), G_l(G_i(k))) - G_i(G_i(k)), 1 - G_l(G_i(k))) \cdot \\
(\beta f_1(G_i(k), G_l(G_i(k))) + (1-\beta) g_{i1}(G_i(k), G_i(k))),
\end{gathered}
$$

or, more compactly,

$$u_1(c, 1-l) = \delta u_1(c', 1-l')(\beta f_1(k', l') + (1-\beta) g_{i1}(k', k')).$$

Having derived these conditions for both the planner and the competitive consumer, how do they differ? To understand why the planning allocation cannot be the same as the competitive allocation, suppose that they were, that is, suppose $h_i = G_i$ and $h_l = G_l$. The first-order conditions for leisure then look identical. The generalized Euler equation, however, does not: it looks identical in all places except for the terms involving how the future self reacts to more current savings. In the planning problem, this effect is $h_i'(h_i(k))$; in the competitive solution, it is $g_{i1}(G_i(k), G_i(k))$, which would equal $g_{i1}(h_i(k), h_i(k))$ if the

[8] Note that these effects would vanish were the consumer not to realize that his future self would deviate from his current plan.

allocations coincided. But they will not, since, for any k, $h_i'(k)$ would have to equal $G_i'(k)$ which, in turn, equals $g_{i1}(k,k) + g_{i2}(k,k)$. That is, h_i' can only equal g_{i1} if g_{i2} is identically zero.

This argument is general: in any economy where the aggregate capital stock separately affects the competitive savings of the individual consumer, which it does through prices, the two allocations differ. In our particular economy, we know that g_{i2} is negative, and this turns out to be true in calibrated versions of the neoclassical growth model as well (of the kind analyzed in the real-business-cycle literature).

To understand the direction of the effect, observe that since g_{i2} is negative, the planner sees a lower effect on future savings of additional savings today: he sees that a given increase in future capital does not lead to a proportional increase in future income, since there are decreasing returns to capital. Seeing a lower than proportional increase in future income, the planner will save somewhat less than the competitive equilibrium, if $\beta < 1$. With $\beta < 1$, a future income increase is now perceived as beneficial since the next self does not save enough in this case, and with a larger income, he will be induced to increase his savings. As this benefit perceived by the planner is lower than the benefit perceived by each competitive consumer, the planner saves less than does the competitive equilibrium, and both save less than what would be best for them. Moreover, the higher savings of the future selves is an important benefit. Therefore, the competitive equilibrium enjoys higher utility.

If $\beta > 1$, $1 - \beta$ is negative and the planner saves more than the competitive equilibrium, since future income is perceived as harmful: here, the future selves save too much, and since the planner sees the decreasing returns from additional savings, he saves a little extra as these decreasing returns are "good". Furthermore, since current consumers would like to see a lower future savings rate, the competitive equilibrium yet again results in higher utility.

Finally, how can the ranking of labor effort across the competitive and central planning allocations be understood? Consumption and leisure are both normal goods with the present preferences, and since savings are higher in the competitive equilibrium, current consumption is lower. Current consumption goes hand in hand with leisure, so less leisure is consumed as well. This explains why the agents in the competitive economy work harder than the agents in the command economy.

5 Policy Analysis

We now turn to the analysis of equilibrium tax policy. Again, suppose that taxes are set by a benevolent government, one which cannot commit its future behavior—that is, future tax rates cannot be committed to—and which rationally foresees how its future self will set taxes. In other words, we are looking for a time-consistent policy equilibrium in tax rates. This equilibrium is more involved than the one above, since it has an additional equilibrium layer: it requires not only that consumers and firms are in equilibrium, given the equilibrium tax rates, but that these tax rates satisfy a time-consistent maximization problem solved by the government. Conceptually, what makes this harder is that to support a given equilibrium, one needs to compute equilibria with other tax rates (rates that will not be chosen in equilibrium): otherwise, it would not be possible to claim that the government maximizes on the equilibrium path.

Before describing and computing time-consistent policy equilibria, let us briefly report what full-commitment policies would look like in this environment.

5.1 Full-commitment policy

We simply report the results (they are straightforward to derive). In all periods following the current one, the tax rates are

$$\tau_k' = \delta\,(1-\beta) \quad \text{and} \quad \tau_i' = \beta - 1.$$

That is, there is a subsidy to investment in future periods (with $\beta < 1$) to induce the time-inconsistent consumers to save the right amount. This subsidy is financed by a capital tax. The labor tax is zero: the labor-leisure choice is static and will not be distorted with this tax scheme. These tax choices produce the law of motion

$$G\left(\bar{k}\right) = \alpha\delta A\bar{k}^{\alpha}\bar{l}^{1-\alpha},$$

with $\bar{l}$ given by

$$\bar{l} = \frac{(1-\alpha)\lambda}{(1-\alpha)\lambda + (1-\lambda)(1-\alpha\delta)}.$$

Working back to the first period, it is possible to show that all tax rates are zero. Given that the government shares the views of the current agent and that future agents behave as they should, there is no need to distort current behavior. The law of motion in period zero becomes

$$G^0\left(\bar{k}\right) = \frac{\alpha\beta\delta}{1-\alpha\delta\,(1-\beta)}A\bar{k}^{\alpha}(\bar{l}^0)^{1-\alpha},$$

where

$$\bar{l}^0 = \frac{(1-\alpha)\lambda(1-\alpha\delta(1-\beta))}{(1-\alpha)\lambda(1-\alpha\delta(1-\beta)) + (1-\lambda)(1-\alpha\delta)} < \bar{l}.$$

This nonconstant sequence of tax rates, savings rates, and labor choices is a Ramsey allocation of sorts: it maximizes the consumer welfare, and coincides with the choice of a consumer with commitment and direct command of resources.

5.2 Time-consistent policy: equilibrium definition

We now define a time-consistent policy equilibrium. The definition, again simply expressing a subgame perfect equilibrium which is first-order Markov, is borrowed from Krusell and Ríos-Rull (1999). We only treat the case where the government is free to choose all tax rates; equilibria for restricted constitutions are defined similarly.

Definition 2 *A time-consistent policy equilibrium is defined in two parts: a listing of the equilibrium elements and a listing of the properties these elements must satisfy.*

OUTCOMES: The behavior on the equilibrium path ("outcomes") is as follows:

- *Tax outcomes are given by a function* $\tau(\bar{k}) = \left(\tau_k(\bar{k}), \tau_l(\bar{k}), \tau_i(\bar{k})\right).$

- *The law of motion for aggregate capital is given by a function* $G_i\left(\bar{k}\right)$ *and the aggregate labor choice by a function* $G_l\left(\bar{k}\right)$.

- *Prices are given by the functions* $r(\bar{k})$ *and* $w(\bar{k})$.

- *The individual's decision rule for capital is given by the function* $g_i(k,\bar{k})$ *and that for labor by the function* $g_l(k,\bar{k})$.

ONE-PERIOD DEVIATIONS: The one-period deviations to tax rates $\tilde{\tau}=(\tilde{\tau}_k,\tilde{\tau}_l,\tilde{\tau}_i)$ *for the current period, with future taxes given by the tax outcome functions evaluated at the capital stocks implied by the current tax rates and the implied capital accumulation, lead to the following equilibrium responses:*

- $\widetilde{G}_i\left(\bar{k},\tilde{\tau}\right)$ *and* $\widetilde{G}_l\left(\bar{k},\tilde{\tau}\right)$ *describe the law of motion for aggregate capital and the aggregate labor function, respectively, for the one-period deviation to* $\tilde{\tau}$.

- $\tilde{r}(\bar{k})$ *and* $\tilde{w}(\bar{k})$ *describe the pricing functions for the one-period deviation.*

- $\tilde{g}_i(k,\bar{k},\tilde{\tau})$ *and* $\tilde{g}_l(k,\bar{k},\tilde{\tau})$ *describe the individual's decision rules for the one-period deviation.*

The equilibrium elements above must satisfy:

1. *Individual optimization:* $\tilde{g}_i(k,\bar{k},\tilde{\tau})$ *and* $\tilde{g}_l(k,\bar{k},\tilde{\tau})$ *solve*

$$\max_{k',\,l}\left[\begin{array}{c}\lambda\log\left(\tilde{r}(\bar{k})k\left(1-\tilde{\tau}_k\right)+\tilde{w}(\bar{k})l\left(1-\tilde{\tau}_l\right)-k'\left(1+\tilde{\tau}_i\right)\right)+(1-\lambda)\log\left(1-l\right)\\ +\beta\delta V\left(k',\tilde{G}_i(\bar{k},\tilde{\tau})\right)\end{array}\right]$$

$$\equiv\widetilde{V}_0(k,\bar{k},\tilde{\tau}),$$

   where $V\left(k,\bar{k}\right)$ *satisfies*

$$V\left(k,\bar{k}\right)=\lambda\log\left(r(\bar{k})k\left(1-\tau_k(\bar{k})\right)+w(\bar{k})g_l(k,\bar{k})\left(1-\tau_l(\bar{k})\right)-\right.$$

$$\left.g_i(k,\bar{k})\left(1+\tau_i(\bar{k})\right)\right)+(1-\lambda)\log\left(1-g_l(k,\bar{k})\right)+\delta V\left(g_i(k,\bar{k}),G_i(\bar{k})\right).$$

   Note that these requirements imply, as a special case, that $\tilde{g}_i(k,\bar{k},\tau(\bar{k}))=g_i(k,\bar{k})$ *and* $\tilde{g}_l(k,\bar{k},\tau(\bar{k}))=g_l(k,\bar{k})$.

2. *Consistency between individual and aggregate actions:* $\tilde{g}_i(\bar{k},\bar{k},\tilde{\tau})=\tilde{G}_i\left(\bar{k},\tilde{\tau}\right)$ *and* $\tilde{g}_l(\bar{k},\bar{k},\tilde{\tau})=\tilde{G}_l\left(\bar{k},\tilde{\tau}\right)$, *which imply, as a special case, that* $g_i(\bar{k},\bar{k})=G_i\left(\bar{k}\right)$ *and that* $g_l(\bar{k},\bar{k})=G_l\left(\bar{k}\right)$.

3. *Competitive pricing:* $r(\bar{k})=f_1(\bar{k},G_l(\bar{k}))$, $w(\bar{k})=f_2(\bar{k},G_l(\bar{k}))$, $\tilde{r}(\bar{k})=f_1(\bar{k},\widetilde{G}_l(\bar{k},\tilde{\tau}))$, *and* $\tilde{w}(\bar{k})=f_2(\bar{k},\widetilde{G}_l(\bar{k},\tilde{\tau}))$.

4. The government maximizes: $\tau(\bar{k}) = \left(\tau_i(\bar{k}), \tau_k(\bar{k}), \tau_l(\bar{k})\right)$ *solves the following problem:*

$$\max_{(\tilde{\tau}_y, \tilde{\tau}_i)} \widetilde{V}_0(\bar{k}, \bar{k}, \tilde{\tau})$$

subject to:

$$-\tilde{G}_i\left(\bar{k}, \tilde{\tau}\right)\tilde{\tau}_i = \alpha A \bar{k}^{\alpha} G_l\left(\bar{k}, \tilde{\tau}\right)^{1-\alpha} \tilde{\tau}_k + (1-\alpha) A \bar{k}^{\alpha} G_l\left(\bar{k}, \tilde{\tau}\right)^{1-\alpha} \tilde{\tau}_l.$$

This definition does not contain any statements about refinement within the set of Markov equilibria. As for the solutions to the dynamic games the competitive agent and the central planner play in the previous sections, we also require, in addition to the above definition, that the equilibrium be a limit of finite-horizon equilibria. This implies uniqueness in our case.

We now turn to the analysis of different tax constitutions. Throughout, the interpretation of our results focuses on the case $\beta < 1$. We will first discuss positive properties of the equilibria. At the end, we make welfare comparisons.

5.3 Preliminaries

Throughout, the time-consistent policy equilibria we derive have a very simple property: equilibrium tax functions are constant. That is, choices of current tax rates do not depend on the aggregate capital stock:

$$\tau(\bar{k}) = \left(\tau_k(\bar{k}), \tau_l(\bar{k}), \tau_i(\bar{k})\right) = (\tau_k, \tau_i, \tau_l) \quad \forall \bar{k}.$$

This simplifies the government's problem substantially. In choosing the current tax rates, the government, in general, has to take into account how its current tax rates influence future tax choices, via the effect of the current tax choice on capital accumulation and, in turn, the effect of this capital accumulation on next period's tax choices. All these effects can now be ignored, since the last part of this logical chain of effects is broken.[9] Our study of optimal, time-consistent taxation here is significantly more involved than our earlier work in Krusell, Kuruşçu, and Smith (1999). The reason is that, in a framework with separate taxes on capital and labor, the government's choice of current taxes depends on what tax rates are used in the future. This means that it is necessary to solve a fixed-point problem in tax rates. The dependence of current on future taxes is absent when labor is supplied exogenously.

We state two useful results. It is possible to show that current savings, given that future taxes are given by a vector τ and current taxes by a vector $\tilde{\tau}$, satisfy

$$\overline{k}' = \frac{\alpha\delta\beta}{1-\delta(1-\beta)} \times \frac{1-\tau_k}{\alpha(1-\tau_k)+(1-\alpha)(1-\tau_l)} \times \frac{\alpha(1-\tilde{\tau}_k)+(1-\alpha)(1-\tilde{\tau}_l)}{1+\tilde{\tau}_i} A \overline{k}^{\alpha} \tilde{\overline{l}}^{1-\alpha}.$$

[9] It is possible to solve for equilibria in which this link is active using numerical methods, however. See Krusell and Ríos-Rull (1999) for an application in a related context.

This expression corresponds to the function $\tilde{G}_i(\bar{k}, \tilde{\tau})$ in the equilibrium definition above, with the qualification that labor has not been substituted in as a function of capital and future taxes; the current labor supply is simply denoted $\tilde{\bar{l}}$ in the expression. This expression highlights the direct effect of any changes in current tax rates that the government would contemplate. The effects on the current savings rate out of total income is proportional to its effect on $\frac{\alpha\left(1-\tilde{\tau}_k\right)+(1-\alpha)\left(1-\tilde{\tau}_l\right)}{1+\tilde{\tau}_i}$.

Turning to the labor choice, $\tilde{\bar{l}}$ is given by

$$G_l\left(\overline{k}, \tilde{\tau}\right) = \frac{(1-\alpha)\left[\lambda(1-\delta)(1+\varphi)+\delta\beta\right](1-\tilde{\tau}_l)}{\alpha(1-\lambda)(1-\delta)(1+\varphi)(1-\tilde{\tau}_k)+(1-\alpha)\left[(1-\delta)(1+\varphi)+\delta\beta\right](1-\tilde{\tau}_l)}.$$

The labor choice also depends on future tax choices, through φ.

From the planning solution, we suspect that the government will want to manipulate savings rate (downward, when $\beta < 1$). However, the government may not be interested in manipulating just the savings rate, but also the labor supply. As we shall see below, it will not distort the labor-leisure choice when it is free to set all tax rates. When the government faces restrictions on the set of tax rates it can use, though, it may want to use the labor tax. From the above expression, it is clear that it cannot affect the labor supply in an income tax system—when $\tau_k = \tau_l$ and $\tilde{\tau}_k = \tilde{\tau}_l$—but in all other cases we look at, it can, and it will.

5.4 Unrestricted tax bases

We begin by considering the case where the government is unrestricted in terms of tax bases: it can tax capital income, labor income, and investment spending.[10]

It is now straightforward, although quite cumbersome, to derive the following proposition.

Proposition 3 *With no restrictions on either τ_i, τ_k, or τ_l, the time-consistent policy equilibrium reproduces the central planning outcome. Moreover, the tax rate on labor income is zero for all $\bar{k}$. The other tax rates are given by*

$$\tau_k = \frac{-\delta^2\beta(1-\alpha)(1-\beta)}{(1-\delta)(1-\alpha\delta(1-\beta))} < (>)\, 0 \text{ if } \beta < (>) 1$$

and

$$\tau_i = \frac{\delta(1-\alpha)(1-\beta)}{1-\delta} > (<)\, 0 \text{ if } \beta < (>) 1$$

for all $\bar{k}$.

Proof: See the appendix.

The intuition for our first result is clear: given enough instruments, the government can behave as if it directly commands resources at the current date. This means that, given enough resources, the government will choose to behave exactly like the central planner in

[10] A tax on consumption is also allowed: it can be expressed as a combination of the other taxes.

the previous section. In particular, the government will tax investment, if $\beta < 1$, in order to slow savings, as it perceives savings to be too high in the zero-tax economy.

Labor taxes are set to zero; the consumption-leisure margin is not distorted, given the savings choice. The labor choice, therefore, does depend indirectly on the savings rate choice: the investment tax decreases capital accumulation, and the capital subsidy (which further decreases next period's capital) reduces labor supply (see equation 5 of Proposition 1: equilibrium labor supply is increasing in $(1-\tau_l)/(1-\tau_k)$).

5.5 Labor income cannot be taxed

It should be clear from the previous result that if the government operates under a constitutional constraint ruling out the taxation of labor income, the resulting allocation is the same as under an unrestricted constitution. The reason is that the government maximizes in the unrestricted case by selecting zero labor taxes and, thus, cannot do better but can achieve the same outcome when labor taxes are ruled out. The proof, hence, is trivial.

Proposition 4 *With no restrictions on either τ_i or τ_k but with the restriction that $\tau_l = 0$, the time-consistent policy equilibrium is the same as when there are no restrictions on any of the tax rates.*

5.6 Investment spending cannot be taxed

Since investment is taxed (or subsidized) when there are no restrictions on tax bases, ruling out investment taxation will result in a different time-consistent policy equilibrium. It is possible to verify the following:

Proposition 5 *With no restrictions on either τ_k or τ_l, but with the restriction that $\tau_i = 0$, the time-consistent policy equilibrium has $\tau_l > (<) 0$ and $\tau_k < (>) 0$ for $\beta < 1\,(\beta > 1)$. Moreover, the labor choice is lower (higher) than in the laissez-faire equilibrium for $\beta < 1\,(\beta > 1)$.*

The proof, together with closed-form solutions for the tax rates, can be found in the Appendix.

The intuition for the result is as follows. Using the general characterization above of how current savings depend on current and future taxes, note that current savings in the $\tau_i = 0$ case will satisfy

$$\overline{k}' = \frac{\alpha\delta\beta}{1-\delta(1-\beta)} \times (1-\tau_k)A\overline{k}^{\alpha}\tilde{l}^{1-\alpha},$$

since the government budget for this constitution implies $\alpha(1-\tilde{\tau}_k)+(1-\alpha)(1-\tilde{\tau}_l) = 1$ for current taxes and $\alpha(1-\tau_k)+(1-\alpha)(1-\tau_l) = 1$ for future taxes. Therefore, the government cannot directly influence the savings rate under this constitution. Future capital taxes decrease the current savings rate, but they are set by future governments and are therefore outside current control. However, the government can affect the current consumption-leisure bundle—it wants to increase it, which is what it wants to do relative to laissez-faire—and there is one channel available: the labor supply. It can still decrease current labor supply by taxing labor, leading to lower current resources saved for the future. Less resources transferred to the future can also be shown to correspond to a higher value for the current Cobb-Douglas consumption-leisure bundle. Therefore, labor taxes are positive here, and capital taxes negative.

Given that capital taxes are negative, savings rates here are *higher* than in the laissez-faire situation. This result is surprising—because the government would currently want to decrease savings if they could—and can be explained by their inability to affect what is the crucial determinant of current savings in this case: future capital taxes. And future capital taxes are negative: being unable to affect their current savings rate, the future governments induce lower labor supply with a tax on labor, and via the budget constraint this leads to a subsidy to capital!

5.7 Capital income cannot be taxed

When capital income cannot be taxed, we obtain yet another equilibrium allocation. This equilibrium—our most complicated case analytically—is harder to characterize: the allocation can be characterized as a solution to a high-order polynomial equation in the tax rate on labor. It turns out that the tax rate on labor can either be positive or negative (both when $\beta < 1$), and that the savings rate can be higher or lower than in the laissez-faire equilibrium.

Faced with no restrictions, the government would like to decrease the savings rate and not distort the labor-leisure choice. Here, as in the case where investment taxation is not allowed, there are opposing forces. Substituting the current budget into the expression for aggregate savings in Section 5.3 and rearranging, it is possible to see that, for any combination of current and future tax rates, the current savings rate $\tilde{s}(\tilde{\tau})$ satisfies:

$$\frac{1-\tilde{s}(\tilde{\tau})}{\alpha(1-\tilde{\tau}_k)+(1-\alpha)(1-\tilde{\tau}_l)} = \frac{1-s}{\alpha(1-\tau_k)+(1-\alpha)(1-\tau_l)},$$

where s denotes the future savings rate. The formula shows that, when the government budget is taken into account, the current savings rate decreases if and only if $\alpha(1-\tilde{\tau}_k) + (1-\alpha)(1-\tilde{\tau}_l)$ increases. With no tax on capital, the latter clearly goes up with a subsidy to labor, which means a tax on investment. However, a subsidy to labor, from the equation determining labor supply in Section 5.3, increases labor supply. This means a decrease in current leisure, and this is an unwanted effect: the government wants current consumption to go up relative to laissez-faire, but it wants current leisure to go up with it. The result of these opposing effects is that, for some parameter values, the government chooses to decrease the savings rate and have a subsidy to labor, whereas for others it chooses to increase the savings rate and have a tax on labor: unlike in the constitution with $\tau_i = 0$, the current consumption-leisure composite is sometimes maximized with $\tau_l < 0$, and sometimes with $\tau_l > 0$.

For this constitution, there may be more than one equilibrium; verifying this involves showing that more than one of the solutions to the first-order condition is in the feasible range and that this tax rate indeed is a maximum, given that this tax rate is expected to be implemented in the future. In our numerical search over a subset of the parameter space (we report on this search in detail in Section 5.10), we found coexisting equilibria in three cases out of 10,000 (in two of these cases, there were three coexisting equilibria and in one case there were two coexisting equilibria). For example, in one case the three labor tax outcomes are either -0.25, 0.44, or 0.73, that is, either a very high positive tax on labor income is possible, or a significant subsidy, or a not so high but positive rate, all depending on expectations about the future.[11] One could also imagine equilibria with tax rates that are not constant—say, two-period cycles—but we did not examine whether such equilibria actually exist.

5.8 Requiring the labor income tax to equal the capital income tax

It is common for capital income to be taxed at the same rate as labor income. Setting $\tau_y \equiv \tau_l = \tau_k$, we obtain yet another allocation, summarized in the following proposition (verification of the proposition is straightforward and is therefore omitted).

Proposition 6 *With income taxation, the time-consistent policy equilibrium leads to*

$$\tau_i = \frac{\delta(1-\beta)(1-\alpha)}{1-\delta+\delta\beta-\alpha\delta\beta}$$

and

$$\tau_y = 1 - \frac{1-\alpha\delta}{1-\delta+\delta\beta-\alpha\delta\beta}\frac{1-\delta(1-\beta)}{1-\alpha\delta(1-\beta)}$$

for all $\bar{k}$. *The law of motion for total capital equals*

$$G\left(\bar{k}\right) = \frac{\beta\delta}{1-\alpha\delta(1-\beta)}\alpha A\bar{k}^{\alpha}\bar{l}^{1-\alpha},$$

where

$$\bar{l} = \frac{\lambda(1-\alpha)(1-\delta+\delta\beta)}{(1-\alpha)(1-\delta+\delta\beta)+\alpha(1-\lambda)(1-\delta)}.$$

Following the same logic as in the previous arguments, the government chooses a lower savings rate by using a tax on investment coupled with a subsidy to income. Notice that although the savings rate, as a fraction of total production, equals that of the central planning outcome, we do not reproduce the central planning outcome. This is due to the fact that labor effort differs: it is higher and, in fact, equals the labor supply of the laissez-faire competitive equilibrium. This results can be seen easily from the expression for labor supply in Section 5.3: current taxes do not affect equilibrium labor supply under the restriction that $\tilde{\tau}_k = \tilde{\tau}_l$, and since future taxes enter in the same way, future taxes will not affect labor supply either. With the same labor supply as in the laissez-faire equilibrium, we obtain higher labor supply here than in the fully restricted taxation outcome, where the subsidy to capital drives a wedge in favor of leisure.

[11] This case occurs when $\delta = 0.9871$, $\beta = 0.3192$, $\lambda = 0.0892$, and $\alpha = 0.7216$.

5.9 More general constitutional rules: partial commitment

Suppose a constitution can contain rules in the form of constant tax rates to apply forever. That is, suppose we could optimally choose tax rates for investment, capital, and labor—that cannot be changed subsequently—to maximize current utility. In this case, it is straightforward to show that the following choices would be best:

$$\tau_l = 0$$

$$\tau_i = -\alpha\delta(1-\beta) < (>)\, 0$$

and

$$\tau_k = \frac{\alpha\delta^2\beta(1-\beta)}{\alpha\delta\beta + (1-\alpha\delta)(1-\delta(1-\beta))} > (<)\, 0$$

for $\beta < (>)\, 1$. These choices would be accompanied by the law of motion

$$G_i\left(\bar{k}\right) = \frac{\alpha\delta\beta}{\alpha\delta\beta + (1-\alpha\delta)(1-\delta(1-\beta))} A\bar{k}^{\alpha}\bar{l}^{1-\alpha},$$

with $\bar{l}$ given by expression 5 of Proposition 1. That is, the best choice would be to subsidize investment (if $\beta < 1$) and to tax capital. This simply reflects the need to increase future savings; at some point, however, such gains are outweighed by losses from increasing current savings—we have an interior solution. Again, the best way of manipulating savings is to use the investment tax; the capital tax is then chosen to balance the budget, and the labor tax is set to zero.

5.10 Welfare comparisons

It is difficult to demonstrate how welfare differs across the different tax constitutions without resorting to numerical computation; welfare, computed in terms of the V_0s, is always a long, nonlinear expression in the parameters, and we did not succeed in deriving analytical results.[12] We therefore searched numerically over our parameter space. Our search was confined to values of β less than one. In particular, we performed a Monte Carlo study consisting of 10,000 randomly drawn parameter configurations. In each replication, each of the parameters δ, β, λ, and α is independently drawn from a uniform distribution on the interval $[0.01, 0.99]$. The results are reported in Table 1 below.[13]

Table 1

[12] It is not necessary to condition welfare comparisons on the level of the capital stock in this model: a higher initial capital stock raises welfare but does not interact with taxes in our class of parametric economies.

[13] For each of the three cases with multiple equilibria, one of these equilibria is arbitrarily selected for the table.

	Rank in terms of initial period welfare (column 1 = highest, column 6 = lowest)					
Number of cases	1	2	3	4	5	6
6,428	3	6	5	4	2	1
2,037	3	5	6	4	2	1
1,043	3	5	4	2	1	6
148	3	4	6	5	2	1
142	3	5	4	6	2	1
70	3	4	5	6	2	1
54	3	6	4	5	2	1
37	3	5	4	2	6	1
11	3	4	5	2	1	6
10	3	5	2	1	4	6
6	3	6	5	2	4	1
4	3	5	6	2	4	1
4	3	5	2	1	6	4
4	3	4	5	2	6	1
1	3	5	2	4	1	6
1	3	6	5	2	1	4

Notes: This table reports the results of a Monte Carlo study consisting of 10,000 randomly drawn parameter configurations. In each replication, each of the parameters δ, β, λ, and α is independently drawn from a uniform distribution on the interval $[0.01, 0.99]$. For each parameter configuration, the six tax constitutions are ranked according to initial period welfare. (In the table, each of the tax constitutions is assigned a number as follows: 1 = unrestricted, 2 = income tax, 3 = partial commitment, 4 = τ_k is zero, 5 = no taxes, and 6 = τ_i is zero.) The first column records the number of times that the ordering specified in that row occurred in the 10,000 replications (the tax constitutions are ranked in decreasing order of initial period welfare). For example, the ordering (3,6,5,4,2,1) occurred 6,428 times in the 10,000 replications.

Sixteen different rankings between the different constitutions materialized; thus, very general rankings are not possible to obtain. However, the results do display some structure.

Partial commitment of course gives the highest utility for all parameter values—by definition, it is the best constant-tax allocation. The second-best constitution, however, is not always the same one; neither is the worst constitution always the same one.

The table does reveal a partial ranking among constitutions: laissez-faire (no taxes) is always better than an income-tax constitution, which is always better than an unrestricted constitution. Among these constitutions, it holds that more restrictions on government behavior results in higher welfare. In this sense, the result from our previous paper is generalized.

The lack of a general ranking is due to the properties of the constitutions where capital taxes and investment taxes, respectively, are not allowed. For example, the constitution where the investment tax cannot be used can be the best constitution, but it can be the worst one, too. In fact, in more than half of our simulations, it was the best constitution, and in about 10% of the cases it was the worst one. Table 2a displays an example of the former outcome and Table 2b one of the latter.

Table 2a

Tax constitution	τ_k	τ_i	τ_l	Savings rate
Partial commitment	0.01754	−0.13216	0.00000	0.09915
$\tau_i = 0$	−0.00172	0.00000	0.00508	0.08773
No taxes	0.00000	0.00000	0.00000	0.08758
$\tau_k = 0$	0.00000	0.04680	−0.01549	0.08367
Income tax	−0.00494	0.05952	−0.00494	0.08307
Unrestricted	−0.00684	0.06152	0.00000	0.08307
Parameters: $\delta = 0.2733$, $\beta = 0.3530$, $\lambda = 0.7046$, $\alpha = 0.7472$				

Note: The tax constitutions are ranked in decreasing order of initial period welfare.

Table 2b

Tax constitution	τ_k	τ_i	τ_l	Savings rate
Partial commitment	0.01760	−0.02317	0.00000	0.02861
No taxes	0.00000	0.00000	0.00000	0.02844
$\tau_k = 0$	0.00000	1.53757	−0.01791	0.01121
Income tax	−0.01774	1.58333	−0.01774	0.01121
Unrestricted	−1.87035	6.28579	0.00000	0.01121
$\tau_i = 0$	−1.68147	0.00000	0.06580	0.07627
Parameters: $\delta = 0.9058$, $\beta = 0.3209$, $\lambda = 0.5748$, $\alpha = 0.0377$				

Note: The tax constitutions are ranked in decreasing order of initial period welfare.

Clearly, in Table 2a, the fact that the savings rate is high in the $\tau_i = 0$ constitution moves welfare up, but in Table 2b, the savings rate is much too high, even significantly higher than in the partial commitment solution.

Finally, Table 2c shows an example where the constitution with $\tau_k = 0$ gives rise to a very high savings rate as well. It is well above the savings rate under partial commitment, but closer to it in absolute value than is the savings rate under unrestricted taxation.

Table 2c

Tax constitution	τ_k	τ_i	τ_l	Savings rate
Partial commitment	0.37387	−0.38834	0.00000	0.70461
No taxes	0.00000	0.00000	0.00000	0.68833
Income tax	−0.51367	0.97242	−0.51367	0.52824
$\tau_k = 0$	0.00000	−0.19106	0.60636	0.85091
Unrestricted	−3.67590	5.09304	0.00000	0.52824
$\tau_i = 0$	−0.26707	0.00000	0.72902	0.87216
Parameters: $\delta = 0.9721$, $\beta = 0.4542$, $\lambda = 0.2237$, $\alpha = 0.7319$				

Note: The tax constitutions are ranked in decreasing order of initial period welfare.

In conclusion: the government in this economy is capable of creating bad outcomes, even though it is benevolent and rational. Unlike in our earlier paper, Krusell, Kuruşçu, and Smith (1999), we cannot say generally that it is good to restrict government choice: it is possible that some freedom to tax actually results in a better outcome than in a no-tax society. In fact, in our random search over the parameter space, this "second-best" result occurred more than half of the time in the following form: the best constitution is to not allow taxation of investment, but to allow labor and capital taxation, and such a constitution improves on a constitution which does not allow any taxation. We do find that unfettered taxation is always worse than no taxation and that income taxation, that is a case where capital income is taxed at the same rate as labor income, is intermediate between the former two in terms of welfare.

6 Conclusions and final remarks

We investigate the properties of taxation when private consumers have quasi-geometric preferences and when the government is subject to the same commitment problem as are private consumers. We show that tax constitutions leaving the government with more freedom to tax may result in worse outcomes, even though the government is benevolent and rational in setting taxes. For some parameter values, the best tax constitution in our model turns out to be laissez-faire, that is, a system where no kind of taxation is allowed. However, it can also be the case that a system with limited taxation powers attains higher welfare. For example, in a large part of our parameter space, it is beneficial to allow the taxation of labor income and of capital income, while not allowing investment taxation. Such a system may give the government incentives to partially correct for the time-inconsistencies in preferences that shape the decisions of private consumers.

Given the results in our earlier paper, Krusell, Kuruşçu, and Smith (1999), one might have conjectured that constitutions allowing direct taxation of investment would always perform the worst. In that paper, we showed that the government would use the investment tax to decrease savings (in the case the time-inconsistency has a bias toward the present), and that such behavior, although well-intended, would leave the consumers worse off. Direct taxation of investment would be the most powerful tool for the government in achieving this goal, and it would therefore be a more dangerous tool. In this paper, however, the government has conflicting goals when there are restrictions on what taxes they can use. Moreover, the restrictions themselves are a form of commitment, which may help increase welfare. Consequently, it turns out that constitutions allowing investment taxation are worse than those ruling it out for some parameter values.

Neither the laissez-faire equilibrium, nor any of the other constitutions based on restricting tax bases, select the best constant savings rate and labor supply. One way of attaining a better outcome in the class of constant-policy economies is to adopt a constitution which writes into stone a set of constant tax rates and then chooses these rates optimally. Such a constitution could alternatively be thought of as the election of someone to government who cares more about future agents than does our representative consumer. This characterization carries over from our earlier paper. Another result that carries over is the comparison between fully flexible taxation and no taxation: the latter is always better in terms of welfare.

The labor tax is not used when a sufficient number of other instruments are available; when they are not, the labor tax is used. When it is used, it may take the form of either a labor subsidy, resulting in higher labor effort than in the competitive equilibrium, or a labor tax. When it is used as part of a general income tax, the labor choice is not distorted. In general, the labor choice is static, and given the consumption-savings choice, it is better not to distort the labor-leisure choice; however, with certain restrictions on constitutions, the consumption-savings choice and the labor-leisure choice become interlinked.

The laissez-faire equilibrium gives a higher level of labor effort than in the equilibrium where the government has a full set of tax instruments: then, the government taxes savings, thus increasing consumption, and higher consumption levels call for a higher consumption of leisure. In the intermediate constitutions, however, taxation can lead to higher labor effort than in the laissez-faire equilibrium since, say, an investment tax creates government revenue that will be paid back to consumers as a subsidy to labor.

Our results indicate that, when consumers' preferences are time-inconsistent, competitive markets perform well, at least relative to a natural central planning problem. Another fundamental mechanism design problem in economics is whether consumption and production decisions can be "separated". Standard theory with time-consistent preferences implies that they can: with separation, as in the competitive framework, we obtain the same allocation as when consumption and production decisions are made by the same agent. In our model with time-inconsistent preferences, separation of consumption and production decisions is strictly better than no separation. No separation could be expressed as follows: instead of renting out his capital and labor to an anonymous firm which pays the market prices for these services, the consumer would use the inputs himself and produce in his backyard with the same technology as used in the market. Under such circumstances, each consumer would behave as in autarky, thus obtaining the central planning allocation—in miniature—and, consequently, be worse off.

Real-world economies have other reasons for taxing than helping consumers overcome commitment problems. We do not model these other reasons here. Our results should be seen as providing supplementary arguments for/against restricting the taxation possibilities in certain directions. For example, if imposing a balanced-budget rule is constitutionally feasible, then perhaps prohibiting capital taxation is also feasible. Would such a constitutional change be good or bad? Our paper helps address this issue, at least to the extent that we believe time-inconsistency of preferences to be a real phenomenon.

Admittedly, our economy is very special in that we use explicit functional-form assumptions. However, the basic argument—that rules, in the form of restricting tax bases, would be beneficial from the point of view of helping us better deal with the time-inconsistency of preferences—is very likely to generalize. Its key ingredient is that individual savings, keeping individual current asset holdings constant, are decreasing in the aggregate capital stock; this feature holds in a broader class of models than the one we consider here, and it includes the typically calibrated growth model upon which the real-business-cycle analysis is based.

We know less about whether the comparison between different kinds of restrictions on tax bases undertaken in this paper is robust to generalizations of our assumptions. In general, extending our analysis will necessitate the use of numerical analysis. Numerical analysis of general-equilibrium models with time-inconsistent agents has not previously been undertaken. In future research, we plan to develop computational tools for this purpose.

References

Barro, Robert (1997), "Myopia and Inconsistency in the Neoclassical Growth Model", NBER Working Paper 6317.

Bernheim, Douglas B., Debraj Ray, and Sevin Yeltekin (1999), "Self-Control, Saving, and the Low Asset Trap", mimeo.

Krusell, Per and José-Víctor Ríos-Rull (1999), "On the Size of U.S. Government: Political Economy in the Neoclassical Growth Model". Forthcoming in the *American Economic Review*.

Krusell, Per, Burhanettin Kuruşçu, and Anthony A. Smith, Jr. (1999), "Equilibrium Welfare and Government Policy with Quasi-Geometric Discounting". Mimeo.

Krusell, Per and Anthony A. Smith, Jr. (1999), "Consumption-Savings Decisions with Quasi-Geometric Discounting". Mimeo.

Laibson, David (1994), "Self-Control and Saving", MIT, mimeo.

Laibson, David (1996), "Hyperbolic Discount Functions, Undersaving, and Savings Policy", NBER Working Paper 5635.

O'Donoghue, T and Matthew Rabin (1999), "Doing it now or later", *American Economic Review*, 89(1), 103–124.

Phelps, Edmund S. and R.A. Pollak (1968), "On Second-best National Saving and Game-equilibrium Growth", *Review of Economic Studies*, 35, 185–199.

Strotz, Robert H. (1956), "Myopia and Inconsistency in Dynamic Utility Maximization", *Review of Economic Studies*, 23, 165–180.

Appendix

Proof of proposition 1: The proof follows by using our guess for the value function, $V\left(k,\bar{k}\right)=a+b\log\bar{k}+c\log\left(k+\varphi\bar{k}\right)$, and our guess for the law of motion for aggregate capital, $G\left(\bar{k}\right)=s\alpha\frac{1-\tau_k}{1+\tau_i}A\bar{k}^{\alpha}l^{1-\alpha}$. Given these, guesses we can solve the agent's problem to obtain $g(k,\bar{k})=\frac{\beta\delta c}{\lambda+\beta\delta c}\cdot\frac{(Rk(1-\tau_k)+wl(1-\tau_l))}{1+\tau_i}-\frac{\lambda\varphi\bar{k}'}{\lambda+\beta\delta c}$. From the first-order condition for the labor choice we obtain $l=\lambda-\frac{(rk(1-\tau_k)-k'(1+\tau_i))(1-\lambda)}{w(1-\tau_l)}$ and $Rk\left(1-\tau_k\right)+wl\left(1-\tau_l\right)-k'\left(1+\tau_i\right)=\lambda\left(Rk\left(1-\tau_k\right)+w\left(1-\tau_l\right)-k'\left(1+\tau_i\right)\right)$. Given these, we obtain $g\left(k,\bar{k}\right)=\frac{\beta\delta c}{1+\beta\delta c}\frac{(Rk(1-\tau_k)+w(1-\tau_l))}{1+\tau_i}-\frac{\varphi\bar{k}'}{1+\beta\delta c}$. Using this decision rule we can verify our guess for the value function and obtain $\varphi=\frac{(1-\alpha)(1-\tau_l)}{\alpha(1-s)(1-\tau_k)\bar{l}}$, $b=\frac{\alpha\delta+\alpha\lambda-\alpha\delta\lambda-1}{(1-\delta)(1-\alpha\delta)}$, and $c=\frac{1}{1-\delta}$. Inserting $\varphi=\frac{(1-\alpha)(1-\tau_l)}{\alpha(1-s)(1-\tau_k)\bar{l}}$ into the individual decision rules and setting $g(\bar{k},\bar{k})=G\left(\bar{k}\right)$ (which has to hold in competitive equilibrium), we obtain $s=\frac{\delta\beta}{1-\delta(1-\beta)}$. This gives $\varphi=\frac{(1-\alpha)(1-\delta(1-\beta))(1-\tau_l)}{\alpha(1-\delta)(1-\tau_k)\bar{l}}$. Plugging $g(\bar{k},\bar{k})$ into the condition for labor we also obtain $l=\lambda-\frac{\alpha(1-\lambda)(1-\delta)(1-\tau_k)\bar{l}}{(1-\alpha)(1-\delta(1-\beta))(1-\tau_l)\bar{k}}k$ and $\bar{l}=\frac{\lambda(1-\alpha)(1-\delta+\delta\beta)(1-\tau_l)}{(1-\alpha)(1-\delta+\delta\beta)(1-\tau_l)+\alpha(1-\lambda)(1-\delta)(1-\tau_k)}$.

Proof of proposition 2: The proof proceeds as follows: first we derive the value function of the current self, $V_0\left(k\right)$, given a general law of motion of type $k'=s\alpha Ak^{\alpha}l^{1-\alpha}$. We thus obtain a function of s. We then show that value function is higher at $s_1=\frac{\delta\beta}{1-\delta(1-\beta)}$ than at $s_2=\frac{\delta\beta}{1-\alpha\delta(1-\beta)}$.

$$\begin{aligned}
V_0\left(k\right) &= \log\left(\left(1-s\alpha\right)Ak^{\alpha}l^{1-\alpha}\right)-\left(1-\lambda\right)\log\left(w\right)\\
&\quad+\beta\delta\left[\log\left(\left(1-s\alpha\right)Ak'^{\alpha}l'^{1-\alpha}\right)-\left(1-\lambda\right)\log\left(w'\right)\right]\\
&\quad+\beta\delta^2\left[\log\left(\left(1-s\alpha\right)Ak''^{\alpha}l''^{1-\alpha}\right)-\left(1-\lambda\right)\log\left(w''\right)\right]+...
\end{aligned}$$

Given the law of motion for aggregate capital, we obtain

$$\begin{aligned}
V_0\left(k\right) &= \left(\alpha\lambda-1+\frac{\lambda\alpha^2\delta\beta}{1-\alpha\delta}-\frac{\delta\beta}{1-\delta}\right)\log\left(1-\alpha+\alpha\left(1-\lambda\right)\left(1-s\right)\right)\\
&\quad+\frac{1-\delta\left(1-\beta\right)}{1-\delta}\log\left(1-s\alpha\right)+\frac{\lambda\alpha\delta\beta}{\left(1-\delta\right)\left(1-\alpha\delta\right)}\log\left(s\right)+...
\end{aligned}$$

Using the first-order conditions, we can show that the maximizer for the problem above is $s^*=\frac{\delta\beta}{\alpha\delta\beta+(1-\alpha\delta)(1-\delta(1-\beta))}$. We see that $s_2<s_1<s^*$ for $\beta<1$, and that $s_2>s_1>s^*$ for $\beta>1$. Showing the strict concavity of $V_0\left(k\right)$ is the last step of the proof, and we omit it for brevity.

Proof of proposition 3: Given the results of Proposition 1, it is straightforward to derive the following decision rules and aggregate rules for the one-period deviation equilibria:

1. $g_i\left(k,\overline{k},\widetilde{\tau}\right)=\frac{\beta\delta}{1-\delta(1-\beta)}\frac{r(\bar{k})k\left(1-\widetilde{\tau}_k\right)+w\left(1-\widetilde{\tau}_l\right)}{1+\widetilde{\tau}_i}-\frac{(1-\delta)\varphi G_i\left(\overline{k},\widetilde{\tau}\right)}{1-\delta(1-\beta)}$

2. $G_i\left(\overline{k},\widetilde{\tau}\right)=\frac{\delta\beta}{\lambda(1-\delta)(1+\varphi)+\delta\beta}\frac{\alpha\left(1-\widetilde{\tau}_k\right)+(1-\alpha)\left(1-\widetilde{\tau}_l\right)}{1+\widetilde{\tau}_i}A\overline{k}^{\alpha}\left[G_l\left(\overline{k},\widetilde{\tau}\right)\right]^{1-\alpha}$

3. $g_l\left(k,\overline{k},\widetilde{\tau}\right)=\lambda-\frac{\left(rk\left(1-\widetilde{\tau}_k\right)-g_i\left(k,\overline{k},\widetilde{\tau}\right)\left(1+\widetilde{\tau}_i\right)\right)(1-\lambda)}{w\left(1-\widetilde{\tau}_l\right)}$

4. $G_l\left(\overline{k},\widetilde{\tau}\right)=\frac{(1-\alpha)[\lambda(1-\delta)(1+\varphi)+\delta\beta]\left(1-\widetilde{\tau}_l\right)}{\alpha(1-\lambda)(1-\delta)(1+\varphi)\left(1-\widetilde{\tau}_k\right)+(1-\alpha)[(1-\delta)(1+\varphi)+\delta\beta]\left(1-\widetilde{\tau}_l\right)}.$

Here, φ is given by

$$\varphi=\frac{(1-\alpha)\left(1-\delta\left(1-\beta\right)\right)\left(1-\tau_l\right)+\alpha\left(1-\lambda\right)\left(1-\delta\right)\left(1-\tau_k\right)}{\alpha\lambda\left(1-\delta\right)\left(1-\tau_k\right)},$$

where the tax rates are those applying two periods hence and subsequently.

The government's objective is:

$$\max_{\tilde{\tau}_k,\,\tilde{\tau}_l,\,\tilde{\tau}_i} V_0(\bar{k},\bar{k},\tilde{\tau})\equiv$$
$$\lambda\log\left(r(\bar{k})\bar{k}\left(1-\tilde{\tau}_k\right)+w(\bar{k})\bar{l}\left(1-\tilde{\tau}_l\right)-\tilde{G}_i\left(\bar{k},\tilde{\tau}\right)\left(1+\tilde{\tau}_i\right)\right)+\left(1-\lambda\right)\log\left(1-\bar{l}\right)$$
$$+\beta\delta V\left(\tilde{G}_i\left(\bar{k},\tilde{\tau}\right),\tilde{G}_i\left(\bar{k},\tilde{\tau}\right)\right).$$

Using the first-order condition for the labor choice and the expression for $V\left(\tilde{G}_i\left(\bar{k},\tilde{\tau}\right),\tilde{G}_i\left(\bar{k},\tilde{\tau}\right)\right)$ from Proposition 1, we obtain

$$\max_{\tilde{\tau}_k,\,\tilde{\tau}_l,\,\tilde{\tau}_i} V_0(\bar{k},\bar{k},\tilde{\tau})\equiv$$
$$\log\left(r(\bar{k})\bar{k}\left(1-\tilde{\tau}_k\right)+w(\bar{k})\bar{l}\left(1-\tilde{\tau}_l\right)-\tilde{G}_i\left(\bar{k},\tilde{\tau}\right)\left(1+\tilde{\tau}_i\right)\right)$$
$$-\left(1-\lambda\right)\log\left(w\left(1-\tilde{\tau}_l\right)\right)+\beta\delta a+\beta\delta\frac{\alpha\delta+\alpha\lambda-\alpha\delta\lambda-1}{(1-\delta)(1-\alpha\delta)}\log\left(\tilde{G}_i\left(\bar{k},\tilde{\tau}\right)\right)$$
$$+\frac{\beta\delta}{1-\delta}\log\left(\left(1+\varphi\right)\tilde{G}_i\left(\bar{k},\tilde{\tau}\right)\right).$$

Inserting the expression for the law of motion $G_i\left(\overline{k},\widetilde{\tau}\right)$ and ignoring the terms that do not contain current taxes, we obtain

$$\max_{\tilde{\tau}_k,\,\tilde{\tau}_l,\,\tilde{\tau}_i} V_0(\bar{k},\bar{k},\tilde{\tau})\equiv$$
$$\left(1+\frac{\lambda\alpha\delta\beta}{1-\alpha\delta}\right)\log\left(\alpha\left(1-\tilde{\tau}_k\right)+\left(1-\alpha\right)\left(1-\tilde{\tau}_l\right)\right)$$
$$+\left[1-\alpha\lambda+\frac{\lambda\alpha\delta\beta}{1-\alpha\delta}\left(1-\alpha\right)\right]\log\left[G_l\left(\overline{k},\widetilde{\tau}\right)\right]$$
$$-\left(1-\lambda\right)\log\left(1-\tilde{\tau}_l\right)-\frac{\lambda\alpha\delta\beta}{1-\alpha\delta}\log\left(1+\widetilde{\tau}_i\right)$$

Inserting the expression for $G_l\left(\overline{k},\widetilde{\tau}\right)$, we arrive at

$$\max_{\tilde{\tau}_k,\,\tilde{\tau}_l,\,\tilde{\tau}_i} V_0(\bar{k},\bar{k},\tilde{\tau})\equiv$$
$$\left(1+\frac{\lambda\alpha\delta\beta}{1-\alpha\delta}\right)\log\left(\alpha\left(1-\tilde{\tau}_k\right)+\left(1-\alpha\right)\left(1-\tilde{\tau}_l\right)\right)$$
$$-\left[1-\alpha\lambda+\frac{\lambda\alpha\delta\beta}{1-\alpha\delta}\left(1-\alpha\right)\right]\times\log\{\alpha\left(1-\lambda\right)\left(1-\delta\right)\left(1+\varphi\right)\left(1-\widetilde{\tau}_k\right)$$
$$+\left(1-\alpha\right)\left[\left(1-\delta\right)\left(1+\varphi\right)+\delta\beta\right]\left(1-\widetilde{\tau}_l\right)\}+\lambda\left(1-\alpha\right)\left[1+\frac{\alpha\delta\beta}{1-\alpha\delta}\right]\log\left(1-\widetilde{\tau}_l\right)$$
$$-\frac{\lambda\alpha\delta\beta}{1-\alpha\delta}\log\left(1+\widetilde{\tau}_i\right)$$

The government budget constraint is given by:

$$-\frac{\delta\beta}{\lambda(1-\delta)(1+\varphi)+\delta\beta}\frac{\alpha(1-\tilde{\tau}_k)+(1-\alpha)(1-\tilde{\tau}_l)}{1+\tilde{\tau}_i}\tilde{\tau}_i=\alpha\tilde{\tau}_k+(1-\alpha)\tilde{\tau}_l.$$

Substituting for $1+\tilde{\tau}_i$ in the objective function, we obtain the following first-order conditions with respect to $\tilde{\tau}_k$ and $\tilde{\tau}_l$:

$$\begin{aligned}
& -\frac{1}{\alpha(1-\tilde{\tau}_k)+(1-\alpha)(1-\tilde{\tau}_l)}+\left[1-\alpha\lambda+\frac{\lambda\alpha\delta\beta}{1-\alpha\delta}(1-\alpha)\right] \\
& \times\frac{(1-\lambda)(1-\delta)(1+\varphi)}{\alpha(1-\lambda)(1-\delta)(1+\varphi)(1-\tilde{\tau}_k)+(1-\alpha)[(1-\delta)(1+\varphi)+\delta\beta](1-\tilde{\tau}_l)} \\
& +\frac{\lambda\alpha\delta\beta}{1-\alpha\delta}\frac{\lambda(1-\delta)(1+\varphi)}{\lambda(1-\delta)(1+\varphi)[\alpha\tilde{\tau}_k+(1-\alpha)\tilde{\tau}_l]+\delta\beta} \\
= & \ 0
\end{aligned}$$

and

$$\begin{aligned}
& -\frac{1}{\alpha(1-\tilde{\tau}_k)+(1-\alpha)(1-\tilde{\tau}_l)}+\left[1-\alpha\lambda+\frac{\lambda\alpha\delta\beta}{1-\alpha\delta}(1-\alpha)\right] \\
& \times\frac{(1-\delta)(1+\varphi)+\delta\beta}{\alpha(1-\lambda)(1-\delta)(1+\varphi)(1-\tilde{\tau}_k)+(1-\alpha)[(1-\delta)(1+\varphi)+\delta\beta](1-\tilde{\tau}_l)} \\
& +\frac{\lambda\alpha\delta\beta}{1-\alpha\delta}\frac{\lambda(1-\delta)(1+\varphi)}{\lambda(1-\delta)(1+\varphi)[\alpha\tilde{\tau}_k+(1-\alpha)\tilde{\tau}_l]+\delta\beta} \\
& -\lambda\left[1+\frac{\alpha\delta\beta}{1-\alpha\delta}\right]\frac{1}{1-\tilde{\tau}_l} \\
= & \ 0
\end{aligned}$$

Time consistency requires $\tilde{\tau}=\tau$. That leaves three equations (including the government budget constraint) and three unknowns. Note that φ is also a function of τ_k and τ_l. It is now straightforward to verify that the tax rates given in the proposition satisfies these three equations.

Proof of proposition 5:

With no restrictions on τ_k and τ_l, but with the restriction that $\tau_i=0$, the tax rates will be given by

$$\tau_l=\frac{\alpha\delta^2\beta(1-\alpha)(1-\beta)}{(1-\delta)(1-\alpha\delta)+\alpha\delta\beta(1-\alpha\delta+\alpha\delta\beta-\delta\beta)}$$

and

$$\tau_k=-\frac{(1-\alpha)}{\alpha}\tau_l$$

To show that these tax rates are indeed a time-consistent equilibrium, note that the government's problem is:

$$\max_{\tilde{\tau}_k,\tilde{\tau}_l}V_0(\bar{k},\bar{k},\tilde{\tau})\equiv$$

$$\left(1+\tfrac{\lambda\alpha\delta\beta}{1-\alpha\delta}\right)\log\left(\alpha\left(1-\tilde{\tau}_k\right)+\left(1-\alpha\right)\left(1-\tilde{\tau}_l\right)\right)-\left[1-\alpha\lambda+\tfrac{\lambda\alpha\delta\beta}{1-\alpha\delta}\left(1-\alpha\right)\right]$$
$$\times\log\left\{\alpha\left(1-\lambda\right)\left(1-\delta\right)\left(1+\varphi\right)\left(1-\tilde{\tau}_k\right)+\left(1-\alpha\right)\left[\left(1-\delta\right)\left(1+\varphi\right)+\delta\beta\right]\left(1-\tilde{\tau}_l\right)\right\}$$
$$+\lambda\left(1-\alpha\right)\left[1+\tfrac{\alpha\delta\beta}{1-\alpha\delta}\right]\log\left(1-\tilde{\tau}_l\right)$$

subject to the government budget constraint

$$\alpha\tilde{\tau}_k+\left(1-\alpha\right)\tilde{\tau}_l=0.$$

Eliminating $\tilde{\tau}_k$ and rearranging, we obtain

$$\max_{\tilde{\tau}_k,\,\tilde{\tau}_l} V_0(\bar{k},\bar{k},\tilde{\tau})\equiv$$

$$\lambda\left(1-\alpha\right)\left[1+\tfrac{\alpha\delta\beta}{1-\alpha\delta}\right]\log\left(1-\tilde{\tau}_l\right)$$
$$-\left[1-\alpha\lambda+\tfrac{\lambda\alpha\delta\beta}{1-\alpha\delta}\left(1-\alpha\right)\right]$$
$$\times\log\left[\left(1-\lambda\right)\left(1-\delta\right)\left(1+\varphi\right)+\left(1-\alpha\right)\left[\lambda\left(1-\delta\right)\left(1+\varphi\right)+\delta\beta\right]\left(1-\tilde{\tau}_l\right)\right].$$

The first-order conditions from this problem imply

$$1-\tilde{\tau}_l=\frac{\lambda\left(1-\alpha\delta+\alpha\delta\beta\right)\left(1-\delta\right)\left(1+\varphi\right)}{\left[\lambda\left(1-\delta\right)\left(1+\varphi\right)+\delta\beta\right]\left(1-\alpha\delta\right)}.$$

Setting $\tilde{\tau}=\tau$, substituting for φ, and using the government budget constraint we can solve for $\tilde{\tau}_l$ and $\tilde{\tau}_k$ and obtain the stated tax rates and characterization in the proposition.