Chapter 3

Origins of the Diversity of Culture Consumption [*]

1 Introduction

Culture consumption is treated differently than the consumption of other goods in most modern societies: it often receives various forms of government support. From the perspective of economics, one must ask why this is the case. One important element of this inquiry is to understand consumer preferences for culture. Here, we believe it to be important to study differences in culture consumption across individuals. In particular, what explains the diversity in the population of the consumption of culture? We argue that culture goods are not like other consumption goods and, especially, that differences in the consumption of culture may be explained by experience: in other words, the taste for culture is in important parts cultivated. In this essay, we propose a theory focusing on this possibility and examine conditions under which culture diversity can arise due to the experience factor.

Our theory contrasts culture with another, generic, good or activity, which does not require taste cultivation in order to be appreciated. We make the model stylized so that culture is the polar opposite in this regard: without any previous experience in culture consumption, current culture consumption is not appreciated at all. However, in all other respects, these goods are symmetric in utility. Thus, one can first

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imagine a static version of the theory where the differences in culture consumption between two consumers with the same preferences and the same constraints are only due to differences in their past consumption of culture: the one with higher experience in culture consumption will choose higher current consumption. Moreover, the effect of a given difference in experience depends on how close substitutes the two goods are, and if the goods are quite close substitutes, experience becomes a very important determinant of the consumption differences between individuals.

In the dynamic model considered, experience is accumulated as a standard capital good: "investment" is represented by current culture consumption, and there is also depreciation, which we assume to be geometric as in standard capital theory. Thus, forward-looking consumers take into account how current culture consumption enhances the future enjoyment of culture consumption. An increase in current culture consumption therefore leads to an induced increase in future culture consumption. How important experience is—how strong the intertemporal complementarities in culture consumption are—for explaining differences in the choices between the culture good and the generic good depends on the substitutability between the goods and also on other features of preferences and of the individual's constraints.

For each possible current value of the individual's culture experience, the model generates an endogenous choice of culture and, residually, of the generic good. An implication of this behavior is an endogenous law of motion for culture experience: a mapping from the level of experience prior to this period into the experience level at the end of this period. The shape of this law of motion is the main focus of this study. It reveals, among other things, to what extent initial experience differences between individuals, and hence culture consumption levels, can persist and possibly be amplified.

One of the main findings here is that significant *long-run* differences in culture consumption can arise between individuals, even if the only difference between these individuals is the initial experience in consuming culture. Formally, the model delivers a law of motion implying multiple steady states. Thus, the model delivers "endogenous" long-run diversity in culture consumption. This occurs if the two goods are relatively close substitutes: over time, then, consumers either move toward complete specialization in the consumption of the generic good, or toward a mix with a significant emphasis on culture. If, on the other hand, the two goods are not close substitutes, then long-run differences in culture consumption can only be explained by fundamental differences in preferences or constraints and not by initial experience: there is a unique steady state, which is reached from all initial conditions.

If the two goods are close substitutes, there might also be a unique steady state with complete specialization on one of the goods. Here, notwithstanding the level of initial experience, the long-run outcome will be the same. Long-run specialization on the culture good will occur if the constraint set—the constraint that binds the consumption of both goods—is sufficiently generous and the goods are close enough substitutes: then current culture consumption can be set quite high and therefore, induce future consumption in a manner which is beneficial even if the initial experience is very low. This can be understood from the perspective of complementarity between present and future consumption: if this complementarity is sufficiently strong, it will lead rational individuals to take advantage of it. Long-run specialization on the generic good, in contrast, results when the constraint is tight, because the complementarity is then not sufficiently powerful. Thus, there is a "scale effect" in culture consumption.

Large *short-run* differences between two individuals in their consumptions of culture can also result from small differences in their initial experience levels. This only occurs if the two goods are close enough substitutes. Formally, this is also a case of multiple steady states but, in addition, the law of motion for the evolution of experience in culture is here discontinuous. In other words, there is a cutoff level of initial culture experience such that the individual is indifferent between a large and a small level of current consumption, where each of these levels then persists over time, and with slightly lower (higher) initial experience, there is a strict preference for the lower (higher) culture accumulation path. The discontinuity appears as a result of an objective function which is not concave when viewed over sequences of culture consumption, despite being concave in consumption at any given moment in time. The source of the nonconcavity is the complementarity between culture consumption at different points in time.

The above results apply if the individual makes the time allocation decisions in a forward-looking manner and with preferences that are "time-consistent". A requirement for time consistency in the case where discounting is stationary, i.e.„ where the individual discounts consumption k periods away in a way that does not depend on what the current time period is, is that discounting is geometric. The assumption of geometric discounting has been standard for a long time. However, experimental evidence has recently cast some doubt on this assumption.[1] In particular, it has been argued that many individuals tend to have a "present-bias", i.e., to discount nearby periods more heavily per unit of time than faraway periods. This implies time-inconsistency of preferences, and a typical description of individuals with these preferences is made in terms of multiple selves: a given individual consists of a sequence of different selves, among whom preferences are conflicting, and these selves then play a dynamic game.[2] In particular, the current self thinks that the future selves are not "forward-looking enough". Motivated by these findings, the present paper also examines how present-biased preferences alter the predictions discussed above. We restrict the analysis to the case with only a finite number of feasible levels for culture capital that the individual can choose; this makes for a simpler analysis than if the domain is continuous.

Time-inconsistency has several implications in the culture accumulation model that the standard model of preferences does not admit. Moreover, these implications are only present if culture is a good featuring taste cultivation. First, a role for "optimism" and "pessimism" appears. More precisely, in the dynamic game there are sometimes multiple equilibria—multiple decision rules, each of which is a Markov-perfect equilibrium—and these equilibria can be ranked in terms of welfare. Thus, the model offers an explanation of differences in culture consumption that cannot be based on observables. Moreover, in this case, there could be a different role for government policy: an appropriate policy could potentially eliminate the bad equilibrium or equilibria. Second, there is another source of long- as well as short-run differences in culture consumption: whereas the model with standard time-consistent preferences generically delivers either one or three steady-state culture consumption levels and, at most, one discontinuity in the decision rule for

[1] See, e.g., the discussion in Laibson (1994).
[2] This kind of formulation was first made in Strotz (1956) and Phelps and Pollak (1968).

culture accumulation, the model with time inconsistency can deliver an equilibrium decision rule with several jumps and more than three steady states. Third, there are parameter configurations for which pure-strategy equilibria do not exist; i.e., culture consumption diversity arises from endogenous uncertainty. The nature of the findings in the model with time-consistent preferences are related to findings in Krusell and Smith (2003a,b) who study time-inconsistent preferences in the context of a consumption-savings problem. These papers find multiplicity, mixed-strategy equilibria in the case with a discrete domain, and jumps in the decision rule. However, it does not deliver a large number of steady states associated with the same equilibrium decision rule. Such an outcome, on the other hand, can be found in Krusell, Martin, and Ríos-Rull (2004), but then in a context of an optimal public policy problem where the time-inconsistency arises from the expectations formation of the private sector.

The model is stylized and abstracts from other determinants of culture consumption viewed to be important, such as the culture consumer's educational level. However, general education can, in part, be viewed as a substitute for experience, so we think that important aspects of the determinants of culture consumption can be captured with our setup.

There are also connections to the literature on addiction (see, e.g., Becker and Murphy, 1988), where the byproduct from current consumption of, say, cigarettes is modeled as having a negative influence on future utility through the accumulation of a stock. There, it is remarked that multiple steady states are possible under some conditions due to similar nonconvexities arising in the present context, where the effect on future utility of current consumption is also present, but has a positive sign. The two models have intertemporal complementarity in common: current consumption of culture/cigarettes encourages future consumption of culture/cigarettes. However, the addiction model emphasizes a negative, and in any case separate, effect on utility that is not considered here. In particular, we assume that if culture consumption is zero, the stock of culture does not at all influence utility. A version of the addiction model was also studied under time-inconsistent preferences but then in a linear-quadratic case where multiplicity or nonexistence was not explored (see Gruber and Köszegi, 2001).

Finally, since we study a dynamic optimization model which can feature a non-concave objective function, there are connections to existing literature exploring such problems, such as Boyer (1978), Orphanides and Zervos (1993), and Skiba (1978); Wirl and Feichtinger (2000,2004) moreover show that also concave problems can have similar features to those reported here.

Section 2 introduces the general setup under the assumption that preferences are time-consistent and contains the results for that case. Section 3 then studies time-inconsistent preferences. Section 4 concludes with some remarks.

2 Time-consistent preferences

In this section, we will analyze a benchmark model where a culture good is viewed as a good where the current enjoyment of the good is higher if this good has been consumed in the past: it is an "experience good". There is one other good assumed not to be an experience good, and the consumer's choice between these two goods at different points in time is then studied. The constraint faced by the consumer in this simple framework is not interpreted as a monetary budget but as a time constraint. Thus, we can think of the two goods as "activities" rather than as regular goods.

2.1 A simple model of culture habits

First, consider a simple static model where the consumer has a choice between two activities, which we can think of as two goods. The consumer has a total time endowment of I units to spend on the two activities/goods: $x + y = I$, where x and y are the two goods. We will think of x as an activity which does not deserve a "culture" label and y as one that does. For example, x could be the total time spent watching Robinson, whereas y is the total time spent watching Lilla Melodifestivalen.

The preference over the two goods is given by $u(x, yk)$, where u is a standard utility function which is strictly concave in its two arguments. We will think of u as being symmetric in its two arguments, and we will later more specifically use the formulation $u(x, yk) = f(x) + f(yk)$. Moreover, k is a "weight" on culture that, in principle, could differ across consumers and therefore explain why some consumers like higher y levels than others. The main contribution of the present analysis is to "endogenize" k by letting k capture the previous experience in consuming the y good. In other words, culture is an "experience good", and k summarizes the "total experience", or the stock of "culture capital". Thus, we will also explicitly describe how k is increasing in the previous consumption of y, and how consumers take into account how more culture consumption increases the future appreciation of culture consumption when deciding between x and y. For this purpose, a dynamic model is needed, and we now move to the description of this model.

Time is assumed to be discrete and the consumer is assumed to live for an infinite number of time periods. Suppose also, as indicated, that we have flow utility given by a function $u(x, yk)$, and that present-value utility is given by

$$\sum_{t=0}^{\infty} \delta^t u(x_t, k_t y_t),$$

where t subscripts denote the period. As also indicated above, we assume that $x_t + y_t = I$, with $x_t \geq 0$ and $y_t \geq 0$, for all t. Finally, we assume that culture capital accumulates according to $k_{t+1} = h(k_t, y_t)$, where h is increasing in its two arguments. Specifically, we assume that

$$k_{t+1} = (1 - d)k_t + by_t,$$

with $d \in [0, 1]$ and $b > 0$. The formulation implies a certain complementarity between culture consumption at different points in time: because k multiplies y in utility and k depends on past ys, high values for past ys encourage a high current y, and vice versa. As we shall see in the context of a specific example, this complementarity may or may not be sufficiently strong to render the objective non-concave.

Thus, a consumer with an initial stock of culture capital equal to k_0 chooses a sequence $\{x_t, y_t, k_{t+1}\}_{t=0}^{\infty}$ satisfying the time endowment constraint and the culture accumulation equation at all times in order to maximize the present-value utility function.

Let us analyze this maximization problem using recursive methods. A variable with a prime denotes the value of this variable next period. Thus, let $y(k, k')$ solve $k' = h(k, y(k, k'))$ for all (k, k'): the function $y(k, k')$ describes the amount of culture consumption now that is consistent with starting with k now and starting next period with k'. Then, the dynamic programming problem reads

$$v(k) = \max_{k' \geq 0} u(I - y(k,k'), y(k,k')k) + \delta v(k').$$

Assuming an interior solution, this leads to

$$-u_1 y_2 + k u_2 y_2 + \delta v'(k') = 0,$$

and, from the envelope theorem,

$$v'(k) = -u_1 y_1 + u_2(y_1 k + y) = (-u_1 + u_2 k) y_1 + u_2 y.$$

Thus, we have

$$(-u_1 + k u_2) y_2 + \delta((-u_1' + u_2' k') y_1' + u_2' y') = 0.$$

This amounts to a difference equation which is necessary and sufficient for an optimum if $u(I - y(k,k'), y(k,k')k)$ is concave in (k, k'). Making this assumption, let us look for a steady state. This delivers

$$(-u_1 + k u_2)(y_2 + \delta y_1) + \delta u_2 y = 0.$$

In an interior steady state, provided that $y_2 + \delta y_1 > 0$, we observe that $u_2 k < u_1$, i.e., that the static marginal utility of highbrow culture is lower than that of mass culture. In a static sense, this looks like irrational overconsumption of highbrow culture: the consumer would be better served by reducing the consumption of highbrow culture toward the equalization of marginal utilities. In a dynamic sense, however, and this is the relevant sense, it is precisely rational: the consumption of highbrow culture has another benefit, and one that is realized in the future: it helps build, or maintain, the stock of culture capital, thus increasing the marginal utility of such consumption at future dates. Accordingly, the marginal benefits of consumption of the two kinds of goods are indeed equalized in a steady state.

Let us now suppose that $h(k, y) = (1-d)k + by$, so that in steady state $y = (d/b)k$. In addition, we have that $y(k, k') = (-(1-d)/b)k + (1/b)k'$ so that $y_1 = (-(1 -$

$d)/b) < 0$, $y_2 = 1/b > 0$, and $y_2 + \delta y_1 = (1 - \delta + \delta d)/b > 0$. Finally, assume that $u(x, yk) = x^\alpha + (yk)^\alpha$, with $\alpha \in (0, 1)$. Then, we have

$$\left((I - (d/b)k)^{\alpha-1} - k((d/b)k^2)^{\alpha-1}\right) \frac{1 - (1 - d)\delta}{b} = \frac{\delta d}{b} k((d/b)k^2)^{\alpha-1}.$$

Simplified, this equation gives

$$\frac{Ib}{d} - k = \left(1 + \frac{d\delta}{1 - (1 - d)\delta}\right)^{\frac{1}{\alpha-1}} k^{\frac{2\alpha-1}{\alpha-1}}. \tag{3.1}$$

The following three subsections examine the model in more detail without arriving at a complete characterization. Section 2.1.1 looks at "candidate" steady states, i.e., levels of k satisfying the first-order condition derived above. Section 2.1.2 then studies local dynamics around steady-state candidates based on first-order conditions, thus delivering further insights regarding the possible time paths for culture capital. Finally, Section 2.1.3 shows that global concavity of this problem is met when $\alpha \leq 0.5$ but never otherwise. However, it also analyzes conditions for local sufficiency, thus providing conditions under which a candidate steady state is indeed a steady state if the domain is sufficiently restricted. These three subsections are rather detailed and can be skipped by a reader mainly wishing to see what classes of results are possible. These are described, finally, in Section 2.2, where we show examples of decision rules for culture capital accumulation for different parameter configurations, and where we discuss how these decision rules depend on the parameters.

2.1.1 Steady-state candidates: existence and uniqueness

Still presuming the interior condition to be sufficient, let us investigate the existence and uniqueness of a solution to the steady-state condition: this is a candidate steady state. Notice that the exponent on k on the right-hand side,

$$\frac{2\alpha - 1}{\alpha - 1},$$

is globally decreasing in α and that it equals 1 for $\alpha = 0$, 0 for $\alpha = 0.5$, and $-\infty$ for $\alpha = 1$. Thus, under suitable assumptions on primitives, it is easily shown that if $\alpha \leq 0.5$, since the right-hand side of the steady-state equation is increasing and the left-hand side is decreasing, there exists a unique steady state.

If $\alpha > 0.5$, then the curve defined by the right-hand side of the steady-state relation is downward-sloping and strictly convex in k. This means that if it intersects the curve given by the left-hand side, which is linear in k, there are two intersection points (unless the curves are tangent, which would only occur as a knife-edge case in terms of the parameter space). That is, whenever $\alpha > 0.5$, there are two positive steady states if any steady state exists.

Notice, moreover, that notwithstanding what α is, positive steady-state candidates exist if I is sufficiently large.

Finally, let us consider the possibility that $k = 0$ would be a steady state. This would mean that a corner solution would be obtained; that is, the first-order condition would hold with inequality:

$$(-u_1 + ku_2)y_2 + \delta((-u_1' + u_2'k')y_1' + u_2'y') < 0,$$

evaluated at $k = y = 0$. Given the parametric functional forms assumed above, this cannot be true for $\alpha \leq 0.5$, because in that case, both ku_2 and yu_2 are plus infinity, whereas $-u_1$ is bounded (recall that y_1 and y_2 are constants). However, it is precisely the case that if $\alpha > 0.5$, then the expression is strictly negative for all values of the primitives, since ku_2 and yu_2 are then both zero. Thus, a zero-capital steady state with no culture consumption satisfies the local conditions for optimality, if and only if $\alpha > 0.5$.

In sum, the conclusions are as follows. If $\alpha \leq 0.5$, there is a unique steady-state candidate with positive culture consumption. If $\alpha > 0.5$, then there is always a steady-state candidate with zero culture capital and zero culture consumption. Moreover, if the period budget is sufficiently large, two additional steady-state candidates also exist in the case where $\alpha > 0.5$, each of which has a positive stock of culture capital and positive culture consumption.

It remains to be seen whether these candidates are indeed steady states. For this purpose, global concavity of the maximization problem would be sufficient but, as we shall see, this property is hard to confirm generally. It is also important to discuss dynamics here, and they will be explored in detail in the following sections.

A preliminary hypothesis that emerges so far is: (i) in the case of a unique positive steady state, there is global convergence to it; (ii) in the case of a unique steady state which is zero, there is global convergence to it; and (iii) in the case with three steady states, the middle steady state is unstable and there is convergence to the zero-culture steady state or the steady state with culture specialization, depending on the initial conditions. Thus, case (iii) would express the idea of hysteresis in culture consumption: initial conditions, and not preferences, are crucial for understanding why consumers differ in their consumption of highbrow culture. We shall see that this hypothesis about optimal culture consumption is close to, but not entirely, correct.

2.1.2 Local dynamics around steady states

Defining

$$F(k, k') \equiv u(I - y(k, k'), y(k, k')k),$$

the first-order condition reads

$$F_2(k, k') + \delta F_1(k', k'') = 0.$$

Local dynamics can be analyzed by linearization. Thus, at a steady state, we have

$$F_{12}\hat{k} + (F_{22} + \delta F_{11})\hat{k}' + \delta F_{12}\hat{k}'' = 0,$$

where hats denote deviations from steady state. A linear rule sets $\hat{k}' = \lambda\hat{k}$ for some λ. Thus, we have

$$F_{12} + (F_{22} + \delta F_{11})\lambda + \delta F_{12}\lambda^2 = 0,$$

which can be used to solve for λ. In a slightly simplified form, this equation becomes

$$\frac{1}{\delta} + \frac{F_{22} + \delta F_{11}}{\delta F_{12}}\lambda + \lambda^2 = 0. \tag{3.2}$$

By ocular inspection of this second-order polynomial function, it is clear that λ must have positive roots if $X \equiv \frac{F_{22}+\delta F_{11}}{\delta F_{12}} < 0$, provided that it has roots. Moreover, under this condition, a necessary and sufficient condition for local stability of a steady state (that is, for one root above 1 and one root below 1) is thus given by the condition

$$\frac{1}{\delta} + X + 1 < 0,$$

which just requires that the quadratic function be below zero at $\lambda = 1$. Further, local instability (both roots larger than 1) requires that the function is above zero and decreasing at $\lambda = 1$, i.e., that

$$\frac{1}{\delta} + X + 1 > 0,$$

and that

$$X + 2 < 0.$$

It can also be seen that, if there are roots, one of these roots must be larger than $1/\sqrt{\delta} > 1$. To see this, notice that the quadratic function $1/\delta + X\lambda + \lambda^2$ has its minimum at $X + 2\lambda = 0$. The requirement that there is a root thus says that $1/\delta - 2\lambda^2 + \lambda^2 < 0$, i.e., that $\lambda^2 > 1/\delta$, from which the assertion follows.

Without specific restrictions on F, the stability properties cannot be determined. In our special parametric case, where $F(k, k') = (I - y(k, k'))^\alpha + (y(k, k')k)^\alpha$, where $y(k, k') = (-(1-d)k + k')/b$, it can be shown (see the next section) that $F_{11} < 0$, $F_{12} > 0$, and $F_{22} < 0$, so that indeed $X < 0$ as presumed above.

2.1.3 Concavity and local sufficiency of first-order conditions

We have focused on interior solutions in the above discussions, especially at and around steady-state points. Here, we will first develop the conditions under which there is sufficient concavity to guarantee that these assumptions are met for all points in the domain. Then, we will discuss how to check local sufficiency of first-order conditions.

Global concavity We will focus on the case where, as in the special parametric case, $u(x, yk) = f(x) + f(yk)$, f is strictly concave, and where $h(k, y)$ is linear, so that $y(k, k')$ is linear.

With the notation of the previous subsection, then, $F(k, k') = u(I - y(k, k'), y(k, k')k) =$

$f(I - y(k,k')) + f(y(k,k')k)$. We obtain

$$F_1 = (-u_1 + ku_2)y_1 + u_2 y$$

and

$$F_2 = (-u_1 + ku_2)y_2.$$

Thus,

$$F_{11} = y_1[f''(x)y_1 + f'(yk) + f''(yk)k(ky_1 + y)] + y_1 f'(yk) + yf''(yk)(ky_1 + y),$$

$$F_{12} = y_2[f''(x)y_1 + f'(yk) + f''(yk)k(ky_1 + y)],$$

which is positive in our special case since $y_1 < 0$ and $f'(yk) + f''(yk)yk = \alpha(1 + \alpha - 1)(yk)^{\alpha-1} > 0$, and

$$F_{22} = y_2^2[f''(x) + k^2 f''(yk)].$$

For somewhat shorter expressions, we can rewrite F_{11} as

$$F_{11} = f''(x)y_1^2 + f''(yk)(ky_1 + y)^2 + 2f'(yk)y_1.$$

The condition for concavity is that F_{11} and F_{22} both be negative and that $F_{11}F_{22} - F_{12}^2 > 0$. We see that F_{22} is always negative and that F_{11} is negative as well. After some cancellations, we have for the cross term

$$F_{11}F_{22} - F_{12}^2 = f''(x)f''(yk)y_2^2[y_1^2k^2 + (ky_1 + y)^2] + 2f'(yk)y_1y_2^2[f''(x) + f''(yk)k^2]$$

$$-y_2^2(2y_1k(ky_1 + y)f''(x)f''(yk) + f'(yk)^2 + 2f'(yk)(y_1 f''(x) + k(ky_1 + y)f''(yk)).$$

This expression further simplifies to

$$y_2^2[f''(x)f''(yk)y^2 - f'(yk)^2 + 2f'(yk)y_1(f''(x) + f''(yk)k^2) - 2f'(yk)(y_1 f''(x) + k(ky_1 + y)f''(yk))]$$

$$= y_2^2[f''(x)f''(yk)y^2 - f'(yk)^2 - 2f'(yk)kyf''(yk))].$$

We see that this expression is ambiguous in general, since the middle term is negative, whereas the other terms are positive.

For our special case, let us examine this expression in more detail. It is proportional to $f''(x)f''(yk)y^2 - f'(yk)^2 - 2f'(yk)kyf''(yk))$, which reads

$$\alpha^2(1-\alpha)^2(I-y)^{\alpha-2}(yk)^{\alpha-2}y^2 - \alpha^2(yk)^{2\alpha-2} - 2\alpha^2(\alpha-1)(yk)^{2\alpha-2},$$

which is itself proportional to

$$(1-\alpha)^2(I-y)^{\alpha-2}(yk)^{\alpha-2}y^2 + (1-2\alpha)(yk)^{2\alpha-2}.$$

Here, observe that if $\alpha \leq 0.5$, this expression is strictly positive and concavity is therefore globally verified. Recall that this is also the case where the steady state is unique.

If $\alpha > 0.5$, whether the concavity condition is satisfied depends on the value of k. For any fixed positive values of k and y, it is clear that this condition must be met if α is sufficiently close to 0.5. On the other hand, suppose that we fix α at some value above 0.5 and, for simplicity, suppose that we consider values where $k' = k$, so that y is proportional to k. Then, the expression becomes

$$A(I-y)^{\alpha-2}k^{2(\alpha-1)} - Bk^{4(\alpha-1)},$$

where A and B are positive constants. This expression can be simplified to

$$k^{2(\alpha-1)}\left(A(I-y)^{\alpha-2} - Bk^{2(\alpha-1)}\right).$$

Thus, it is clear that as k becomes closer to 0 (and y thus also becomes closer to 0), this expression must turn negative, since $\alpha < 1$. This implies that when $\alpha > 0.5$, although sufficiently large values for k mean that this expression is positive, global concavity of $F(k, k')$ over its entire domain cannot be established.

Local sufficiency To verify that a steady-state candidate represents consumer maximization *locally*, we need to look at whether the objective

$$F(k, k') + \delta v(k')$$

is concave in k' at the proposed candidate $k = k'$. Using the fact that the envelope theorem

$$v_k(k) = F_1(k, g(k))$$

for all values of k, we have that the derivative of the objective with respect to k' is

$$F_2(k, k') + \delta F_1(k', g(k')).$$

Assuming g to be differentiable, and using the notation $\lambda = g_k$ (consistently with the previous section, λ is the local slope of the decision rule), we obtain that the second derivative of the objective with respect to k' equals

$$F_{22} + \delta F'_{11} + \delta F'_{12}\lambda'.$$

In other words, we see that local concavity is satisfied at a steady state, if this expression is negative, i.e., if

$$\lambda < -\frac{F_{22} + \delta F_{11}}{\delta F_{12}} = -X,$$

assuming that $F_{12} > 0$, which it is in our special case. That is, λ must be lower than X, where λ also solves $1/\delta + X\lambda + \lambda^2 = 0$. As we shall now see, the existence of a positive real root ensures concavity. There is a solution to the polynomial equation if the minimizer of the polynomial, $\lambda = -X/2$ leads to a non-negative value. If there is a solution, the smaller solution must satisfy $\lambda < -X/2$, which is less than $-X$, thus ensuring concavity. Accordingly, we know that if we find positive steady-state candidates, they will satisfy local sufficiency.

2.2 Full characterization of decision rules and comparative statics

In this section, we characterize decision rules for culture accumulation, $g(k)$. Most of the characterization is based on numerical model solution, but the insights obtained parallel those in the previous sections, where candidate steady states and local stability around steady states were examined.

A general feature present in the parametric case of the model considered is that $g(k)$ is globally increasing. The economic content of this feature is that if consumer A starts with a higher culture capital than consumer B, then consumer A will always have a higher culture capital than consumer B. In Section 3, we prove this assertion for a generalized version of our model. This property will be visualized in the decision rules computed and graphed in this section.

We would also like to know whether the amount of culture consumed, i.e., $(g(k) - (1-d)k)/b$, is also globally increasing in k: whether consumers with a higher culture capital stock consume more culture. This feature may hold generally—it is intuitively plausible—but we have not been able to prove it. It is verified in all the examples we have computed numerically. Clearly, it also holds in the special case where there is full depreciation, i.e., when $d = 0$, because then the result follows from $g(k)$ being increasing.

The analysis is divided into two broad sections: the $\alpha \leq 0.5$ case and the $\alpha > 0.5$ case. For each case, we explore the shapes of decision rules and how these decision rules are influenced by the parameters of the model.

2.2.1 Low substitutability: $\alpha \leq 0.5$

Here, many properties of the decision rule can be ascertained based on the above results. From the inspection of equation (3.1), we know that the steady state is unique, and that it satisfies the global conditions for a maximum. Moreover, we can use standard methods to show that $g(k)$ is a continuous function.

Regarding comparative statics, an inspection of this equation reveals that the steady-state level of culture capital increases in Ib and δ. These effects are intuitive: I allows a higher consumption of both goods, and b raises the relative appreciation of the culture good; increased patience makes the consumer place a higher weight on the dynamic benefits of consuming culture, leading to a higher long-run culture capital stock. The effects of an increase in depreciation, d, on k are ambiguous. On the one hand, higher depreciation acts as a lowering of the return to accumulating culture capital—thus working toward a lower culture capital stock. On the other hand, for a given amount of culture capital built for next period and a given starting level of culture capital, an increased depreciation raises the required culture consumption—

$y = (k' - (1-d)k)/b$—and since culture consumption is complementary over time, this increase in y required by the increased need to replenish k induces an increase in k. We will look at how these two effects play out below.

Finally, an increase in α, the substitutability parameter, makes the function of k on the right-hand side of equation (3.1) both shift down (the constant) and become more curved; it becomes steeper at zero and shifts down for high values of k. What is the resulting effect on the steady state? Note that if $k = (1+r)^{-1}$, where $r \equiv \frac{d\delta}{1-\delta+d\delta}$, then the right-hand side equals $(1+r)^{-2}$ for all values of α! I.e., for a specific value of k, the right-hand side does not vary with k, which means that a change in α makes the right-hand side rotate around $((1+r)^{-1}, (1+r)^{-2})$. More precisely, for $k < (1+r)^{-1}$, the function increases in α, and for $k > (1+r)^{-1}$, the function decreases in α. Thus, the steady-state effect of changing α depends on whether at $k = (1+r)^{-1}$ the left-hand side is above or below the right-hand side, i.e., whether $(bI)/d - (1+r)^{-1}$ is above or below $(1+r)^{-2}$. This, among other things, is regulated by I. So if I is sufficiently high, because the left-hand side is above the right-hand side at $k = (1+r)^{-1}$, the steady state is above $(1+r)^{-1}$, and an increase in α will raise the steady state. Conversely, if I is sufficiently low, the steady state is below $(1+r)^{-1}$, and in this case a rise in α will lower the steady state.

The intuition for the comparative statics with respect to α can be phrased in terms of "scale effects": an increase in the substitutability between goods will cause more specialization in one of the goods, and if the time/resource constraint is sufficiently lax, this favors the culture good, because the enjoyment of the culture good over time allows complementarity of resources over time. To see this, suppose that you always set $y = \lambda I$, independently of k. Then, in the long run, k will be $b\lambda I/d$, and the utility enjoyment from culture per period will be $(ky)^\alpha = (b\lambda^2 I^2/d)^\alpha$. In contrast, the enjoyment from the generic good will be $((1-\lambda)I)^\alpha$. That is, since the enjoyment of culture involves the square of I and the enjoyment of the generic good only involves I, we can call this a scale effect. So if the scale is sufficiently large, more substitutability will favor the culture good; otherwise it will favor the generic good.

Figure 3.1 below shows the decision rule for a typical parameter configuration where the scale effect is weak (I is low), and it also shows the decision rules for a

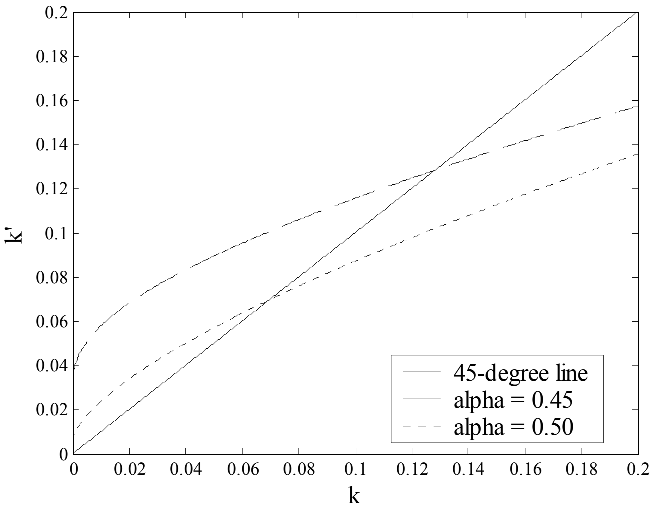

Figure 3.1:

higher and a lower value of α.[3] As substitutability is increased, the figure reveals that the whole decision rule moves down, thereby implying a lower steady-state value for culture capital. The interpretation, as explained above, is one of scale effects in conjunction with substitutability: as there is more substitutability between the goods, the agent is more willing to forgo the culture good in order to focus on the generic good, since the scale is not sufficiently large for the intertemporal culture complementarities to pay off. Thus, culture consumption and accumulation are lower for all values of k.

In the borderline case when $\alpha = 0.5$, the decision rule has the same character as when $\alpha < 0.5$: it has $g(0) > 0$ and it is increasing and concave.

Figure 3.2 below shows the comparative statics in the case where the rate of depreciation is changed; an increase in d decreases cultural accumulation globally, and the steady state falls in particular.

In Figure 3.3, the discount factor is varied, and we see how an increase in the discount factor—implying more patience—globally increases culture capital accumulation and culture consumption.

[3] The parameter values are $\alpha = 0.45$, $I = 1$, $d = 0.75$, $b = 0.25$, and $\delta = 0.96$.

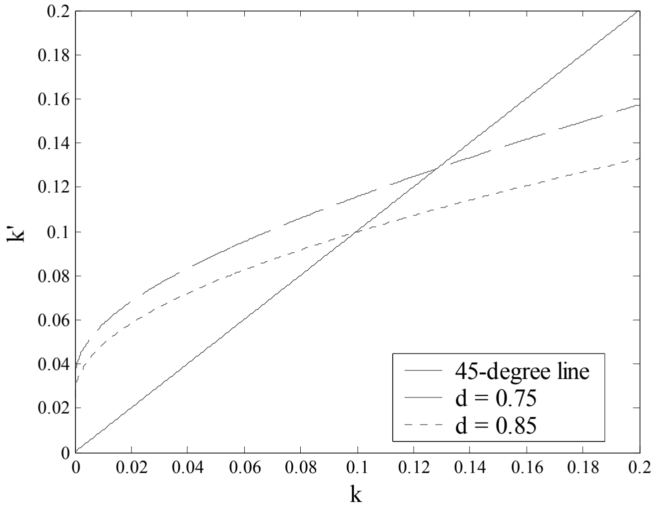

Figure 3.2:

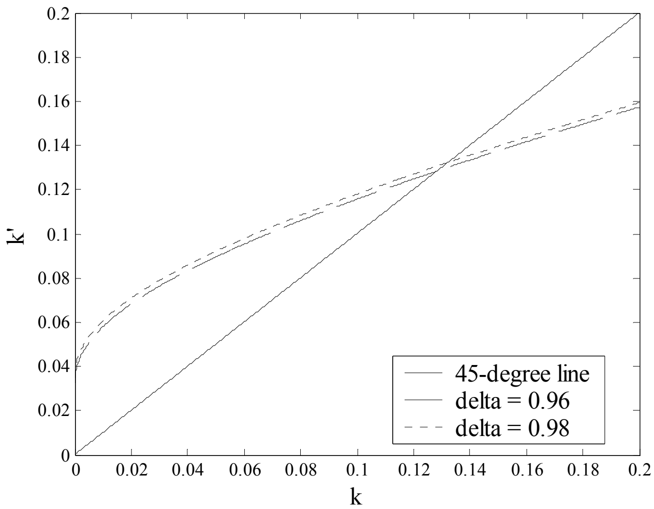

Figure 3.3:

2.2.2 High substitutability: $\alpha > 0.5$

When α is above 0.5, so that substitutability is sufficiently strong to imply (i) that there are two positive steady-state candidates and (ii) that the objective $F(k, k')$ is not globally concave in (k, k'), the decision rules do not only change qualitatively, but there are also different kinds of decision rules.

First, let us look at the set of candidate steady states, i.e., at equation (3.1). The right-hand side is now a convex, downward-sloping function of k. Thus, if it intersects the left-hand side, it does so twice (generically). These two steady states go further apart—the lower steady state decreases and the upper steady state increases—if Ib goes up, because that shifts the left-hand side up. If δ goes up, the right-hand side shifts down, which has a similar effect. Thus, for the higher steady state, the comparative statics is the same as in the case where $\alpha \leq 0.5$, but for the lower steady state, the comparative statics reverse. Similarly, the effects of changes in d are ambiguous.

As for the comparative statics of the candidate steady states with respect to the substitutability parameter α, we have the same features as before: as α changes, the right-hand side of the equation rotates around a given point $((1+r)^{-1}, (1+r)^{-2})$, and the resulting changes in the steady states depend on where this point is relative to the left-hand side at $k = (1+r)^{-1}$. If, say, I, is sufficiently large so that $(bI)/d - (1+r)^{-1} > (1+r)^{-2}$, then an increase in α raises the lower as well as the higher steady state. If I is not sufficiently large, then an increase in α raises the lower steady state and lowers the higher steady state.

Turning to the exploration of local dynamics, we verify numerically by solving equation (3.2) that the higher steady state always has real roots and is dynamically stable: the candidate decision rule slope at that steady state is always positive (and less than one). The lower steady state either yields real roots—implying a slope higher than 1—or delivers complex roots. The latter is always the case, when the lower steady state is close enough to zero for example. Thus, in the cases where there is no real root, the associated candidate steady state cannot be a steady state. Even when a candidate steady state has real roots and thus, from the analysis in Section 2.1.3, has a policy choice which is a local maximizer, it may not be a global maximizer. Our numerical computation of entire decision rules is therefore necessary for a full investigation.

We will consider three typical cases. In one case, the decision rule is continuous and intersects the 45-degree line twice for positive values, where the lower steady state is unstable and the higher steady state stable. In another case, the decision rule has a discontinuous jump at some value for k. In a third case, the decision rule is either everywhere below the 45-degree line or intersects it once. It will, however, be useful to start with a simple extreme example: that where $\alpha = 1$.

Perfect substitutes When the two goods are perfect substitutes, the period utility function is linear, given k. We will show how to solve this simple case analytically, assuming that I is sufficiently large. Guess that the value function $v(k)$ is linear in k: $v(k) = a + ck$, where a and c are scalars, so long as $k \geq k^*$, and that $v(k) = \bar{v}$, where $\bar{v}$ is a scalar, otherwise. In other words, the guess is that $y = I$ if $k \geq k^*$ and that $x = I$, otherwise. It is easily checked that $\bar{v}$ must equal $I/(1-\delta)$.

To verify that the solution is correct and show how to find a, c, and k^*, first suppose that $k > k^*$. We must then have that the dynamic programming equation is satisfied at the conjectured solution, i.e.,

$$a + ck = kI + \delta\{a + c((1-d)k + bI)\}.$$

This must hold for all $k \geq k^*$, leading us to identify the coefficients a and c as

$$a = \frac{\delta cbI}{1-\delta},$$

and

$$c = \frac{I}{1-\delta(1-d)}.$$

The cutoff value k^* is now defined as the value of k making the individual indifferent between choosing $x = I$, which delivers $I/(1-\delta)$, and $y = I$, which delivers $a + ck^*$. Substituting in the solutions for a and c, we thus have

$$\frac{I}{1-\delta} = \frac{\delta cbI}{1-\delta} + \frac{I}{1-\delta(1-d)}k^*,$$

which gives

$$k^* = (1 - \delta cb)\,\frac{1 - \delta(1-d)}{1-\delta} = 1 + \frac{\delta}{1-\delta}(d - bI).$$

If this expression is negative, the set of k for which $x = I$ is chosen is empty.

We now need to verify that the proposed behavior is indeed optimal. For this purpose, notice that the maximization problem reads

$$\max_{y \in [0,I]} I - y + ky + \delta\{a + c((1-d)k + by)\},$$

assuming that $(1-d)k + by \geq k^*$. In this case, it is optimal to set $y = I$ so long as $k - 1 + \delta cb \geq 0$. Thus, we need to show that at $k = k^*$, this inequality is satisfied; if so, it is also satisfied at higher values for k. Inserting $k^* = (1-\delta cb)\,\frac{1-\delta(1-d)}{1-\delta}$, it is easily seen that $k^\star \geq 1 - \delta cb$, with strict inequality whenever $d > 0$, i.e., the inequality is indeed satisfied. We also need to verify that the value of k' is indeed chosen as k exceeds k^*. This is true if $(1-d)k^* + bI \geq k^*$, which (after some little algebra) delivers the restriction

$$bI \geq d. \tag{3.3}$$

Thus, the restriction is met if bI is sufficiently large or d is sufficiently small; this is natural, because it just says that replenishing k is "easy". Notice also that this restriction implies $k^* \leq 1$.

Finally, we need to verify that if $k < k^*$, it is optimal to set $x = I$. The choice $x = I$ always delivers $I/(1-\delta)$, notwithstanding what is k. The choice $y = I$ gives $kI + \delta\{a + c((1-d)k + bI)\}$, if k is sufficiently large so that $k' \geq k^*$, otherwise it gives $kI + \delta I/(1-\delta)$. In the first of these cases, the objective becomes $k(I + \delta c(1-d)) + \delta(a + bcI) = kI/(1-\delta(1-d)) + \delta(a + bcI)$, which is increasing in k and equal to $I/(1-\delta)$ at $k = k^*$; therefore it must be lower for lower values of k. In the second case, $x = I$ is optimal if $k \leq 1$. But we know this to be true since $k^* \leq 1$ from above. This completes the proof.

Notice that in the case covered above, either there is specialization at all times and that the specialization is always the same: either the culture good is never consumed or the generic good is never consumed. In the case where the inequality in (3.3) is not met, the situation is different. The solution now is also to choose $x = I$ below some cutoff value of k and $y = I$ above that value, but in this case

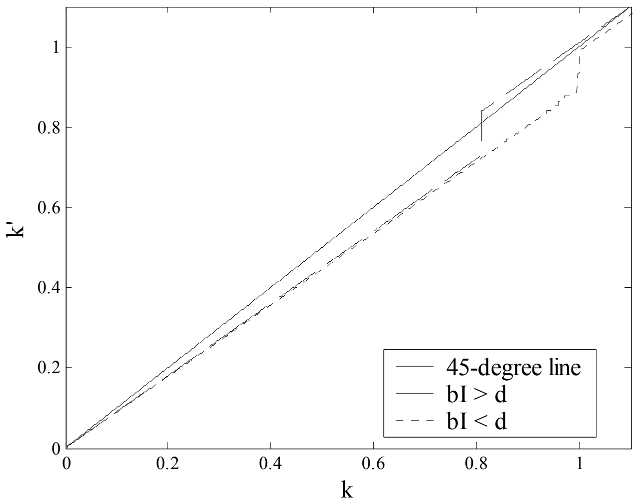

Figure 3.4:

$k' < k$ for all k. Thus, for large values of k, $k' = (1 - d)k + bI < k$, and when k is sufficiently low, $k' = (1 - d)k$. The resulting time series means that the culture good will be consumed for a finite number of periods, after which there is a switch to the generic good, which will then be consumed forever after.[4] Figure 3.4 shows the decision rules for the two cases: one case where the inequality in (3.3) is met and one where it is not.[5]

Case 1: continuous decision rules We now look at the case of a continuous decision rule, where the point at which the right hand side of equation (3.1) pivots is above the left hand side of the equation. Figure 3.5 below displays two decision rules—for different values of α—where the rule "hugs" the 45-degree line.[6] It makes the dynamics of culture accumulation clear: if the initial stock of culture capital is sufficiently large, culture capital will end up at a relatively high level in the long run, but if it is not sufficiently large, culture capital will converge to zero, and this

[4] Showing this formally is somewhat more tedious; the value function is piece-wise linear with a countably infinite number of segments with increasing slopes. The slope for the smallest values of k is zero, as in the first example, then it becomes I, then $I(1 + \delta(1 - d))$, etc, and this sequence converges to the slope of the first example, namely $I/(1 - \delta(1 - d))$.

[5] The parameter values are $\alpha = 1$, $I = 1$, and $d = 0.10$, $b = 0.11$ or $d = 0.11$, $b = 0.10$.

[6] The parameter values are $\alpha = 0.53$, $I = 2.3$, $d = 0.75$, $b = 0.25$ and $\delta = 0.50$.

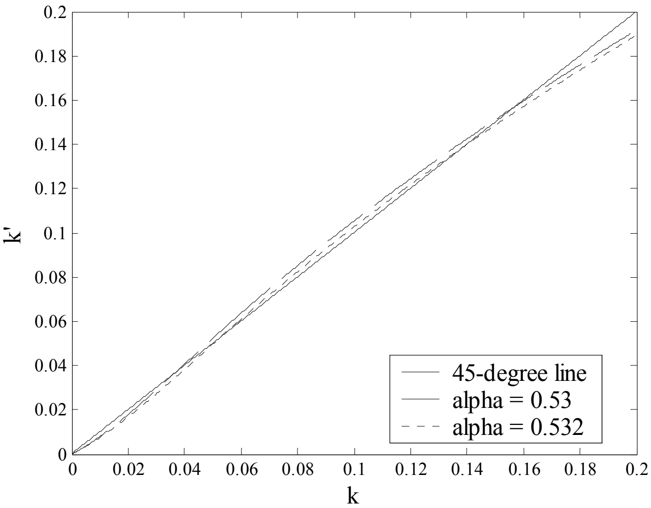

Figure 3.5:

consumer will not consume culture in the long run.

The comparative statics in the picture shows that a higher α lowers the curve, implying that the higher steady state falls and the lower steady state increases, thus implying that the range of initial conditions for which culture consumption is zero in the long run increases; similarly, if culture consumption converges to a positive number, it is a smaller number for a larger value of α. The explanation for this is that I, or bI, is not sufficiently large; recall the discussion of steady-state candidates at the beginning of the present section. Since scale is relevant, when I is not large, more substitutability tends to favor the generic good, which is the intuitive explanation for the comparative statics just noted.

If α is increased enough, the decision rule would fall below the 45-degree line, thus implying that k converges to zero. For $\alpha = 1$, as we know, the decision rule becomes $k' = (1 - d)k$, as previously discussed (unless k is very large).

Figures 3.6 and 3.7 show comparative statics with respect to d and δ. Here, as in the case of $\alpha \leq 0.5$, we see that higher depreciation lowers the culture accumulation rule globally, in this case implying that the range of initial conditions for which k converges to zero increases and that the positive long-run attraction point is lower.

In contrast, and once more as in the case with lower substitutability, stronger

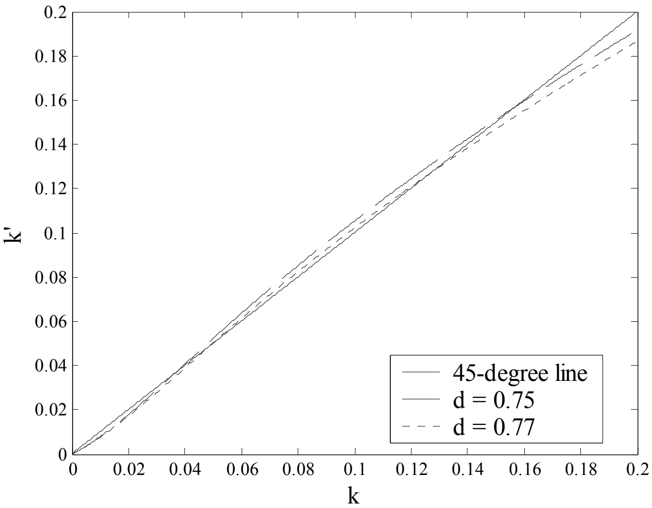

Figure 3.6:

patience raises the culture accumulation rule globally, making the range of initial conditions over which k converges to zero shrink and raising the high steady state.

Case 2: discontinuous decision rules Figures 3.8 depicts a case with discontinuous culture accumulation rules, where the point at which the right hand side of equation (3.1) pivots is above the left hand side of the equation.[7]

Figure 3.8 shows a case where the rule takes a jump over the 45-degree line for the middle value of α, $\alpha = 0.55$. Clearly, at the jump, the individual is indifferent between a high level of culture consumption and accumulation, and a much lower one. With slightly more culture capital, the individual strictly prefers the higher trajectory; with slightly less culture capital, the lower trajectory, which converges to zero, is preferred.

Interestingly, an increase in α eliminates the jump. Here, with more substitutability, although a positive higher steady state exists, it does not correspond to a global maximum. Instead, it is optimal for all values of k to decrease k toward zero over time, leading to zero culture consumption in the long run. If, on the other hand, α is decreased, we see that the jump disappears as well! Here, with more substi-

[7] The parameter values are $\alpha = 0.55$, $I = 10$, $d = 1$, $b = 0.05$, and $\delta = 0.96$.

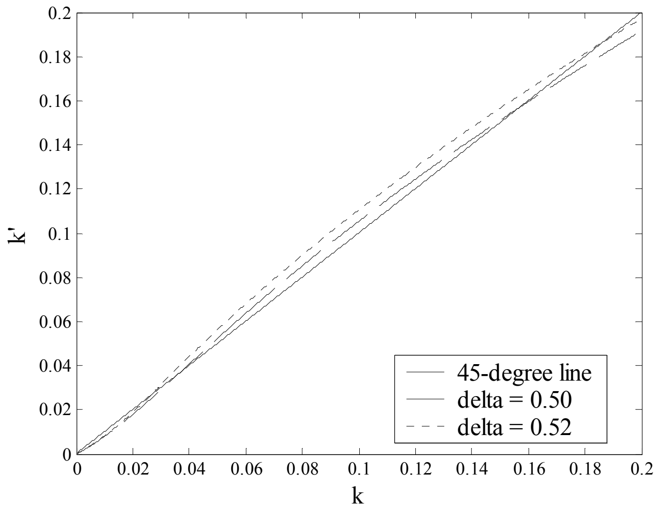

Figure 3.7:

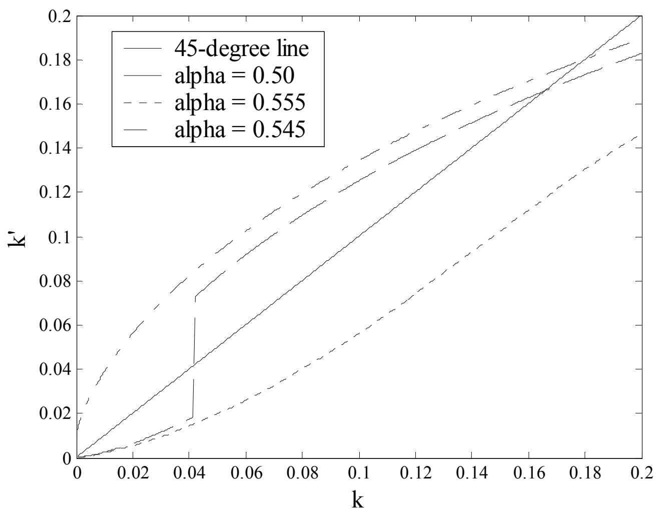

Figure 3.8:

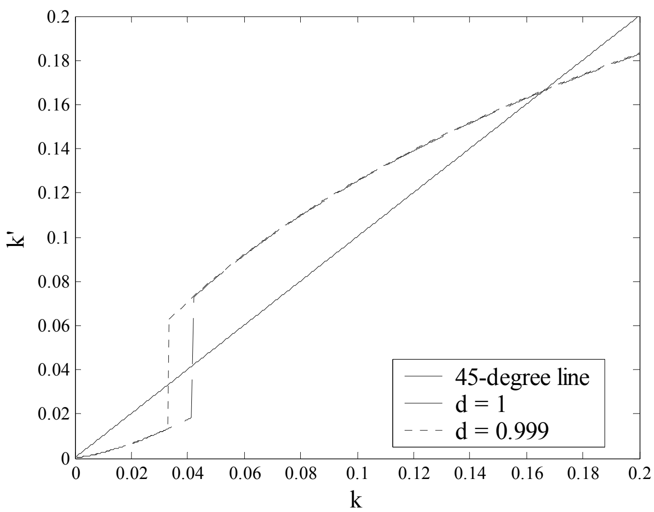

Figure 3.9:

tutability, the behavior looks like the case where $\alpha \leq 0.5$. The interpretation is now that the culture good is more valued, not primarily because of the complementarity in consuming it over time, but because the consumer wants a more balanced mix of the two goods.

Naturally, the changes in α in Figure 3.8 that are less drastic would maintain the jump and move the point of discontinuity to the left (if α decreased) or to the right (if α is increased).

Figure 3.9 shows that a decrease in depreciation raises the decision rule, and moves the discontinuity to the left.

Once more, less depreciation promotes culture consumption, and the range of initial conditions for which culture consumption converges to zero shrinks; relatedly, the positive steady state is higher. The example also shows that the change in the decision rule is very slight at low and high levels of k, whereas the changes implied by the move in the point of discontinuity are drastic.

Figure 3.10, finally, illustrates a case where the decision rule falls, due to a decrease in the degree of patience. As in the previous case, the nonlinearity of the present framework leads to very small effects for small and large values of k, but drastic effects in a middle range. For example, imagine consumers to have the

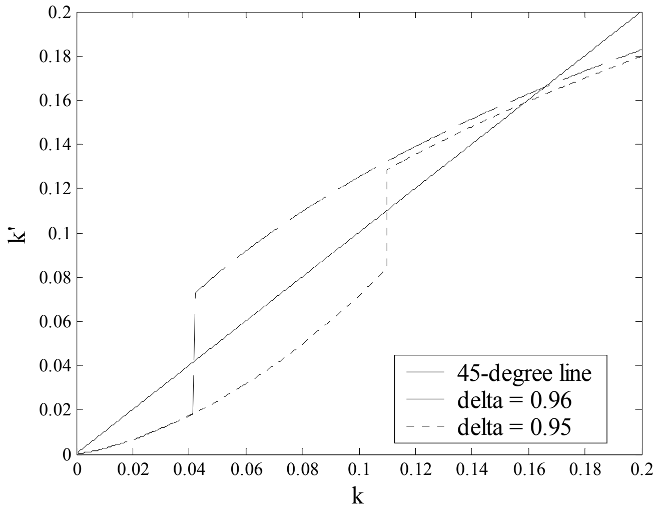

Figure 3.10:

same initial conditions for culture capital but to display differences in patience or "memory" with regard to retaining cultural experience beyond one period. Then, very large short-run and long-run differences between these consumers in culture accumulation can result, even though the differences in preferences are slight.

Case 3: unique steady states and global convergence We have seen that, for $\alpha \in (0.5, 1)$, it is also possible that there is a unique positive steady state. That outcome was shown in Figure 3.8. In that case, there was global convergence to a positive long-run level of culture capital. The same figure also demonstrated that, when one parameter was slightly changed, there would be global convergence to zero long-run culture consumption. The difference was in the degree of substitutability.

2.2.3 Summary and implications

The model has been demonstrated to possess a rich set of qualitative outcomes in terms of its law of motion for culture capital. A key parameter behind the results is α: the curvature of the utility function u, which in this case regulates the substitutability of x and y, given k: the lower the α, the less substitutable are x and

y.[8] The prediction is thus that if highbrow-culture goods and mass-culture goods are close substitutes in a static sense, there can be multiple steady states, and long-run levels of culture consumption can critically depend on the initial conditions. Moreover, short-run levels of culture can also significantly respond to the initial conditions, since the decision rules are sometimes discontinuous. Intuitively, if the goods are not close substitutes, they are both essential and low-culture "traps" are not possible: highbrow culture is simply too important to become a neglected good. If the goods are close substitutes, however, then either highbrow culture is entirely competed out (the zero-culture steady state) or the polar opposite case, namely, with specialization in highbrow culture is also possible: high consumption of these goods and low consumption of mass-culture goods.

We also saw that the other parameters of the model can be of importance. Parameter I—the time endowment, or total resources available to spend on the two goods—can be seen as a scale measure, and when it is large, more specialization on culture goods tends to occur, because culture consumption is subject to a scale effect. Higher depreciation of culture capital led, as might be expected, to lower levels of culture capital, and more patience on the part of the individual led to higher levels of culture capital. Moreover, small differences in these parameters across consumers could imply large short- and/or long-run differences in the level of culture consumption.

3 Time-inconsistent preferences

Suppose now that there is quasi-geometric discounting, regulated by parameter β, which we shall assume to be less than 1. The consumer problem becomes

$$w(k) = \max_{k' \geq 0} u(I - y(k, k'), y(k, k')k) + \beta\delta v(k'),$$

where

$$v(k) = u(I - y(k, g(k)), y(k, g(k))k) + \delta v(g(k))$$

[8] The elasticity of substitution turns out to be $1/(1 - \alpha)$.

and $g(k)$ is the perceived future behavior in the infinite-horizon game: the consumer's behavior at a point in time is assumed to depend on nothing but his current culture capital stock. The behavior given by $g(k)$ is a Markov-perfect equilibrium if $g(k)$ solves the stated maximization problem for all k. In what follows, we will assume $u(x, ky) = x^\alpha + (ky)^\alpha$ and that $h(k, y) = (1 - d)k + by$.

Assuming, as above, that $u(x, yk)$ and $y(k, k')$ are in the special parametric classes, we can establish an important property of an equilibrium: that the decision rule $g(k)$ is increasing, i.e., consumers with larger initial stocks of high-brow culture capital will also have larger such stocks for the remaining time. To see this, we first establish a lemma. Once more, we will use the notation $F(k, k') = u(I - y(k, k'), y(k, k')k)$.

Lemma: *If $k_2 > k_1$ and $k_2' > k_1'$, then $F(k_2, k_2') - F(k_2, k_1') > F(k_1, k_2') - F(k_1, k_1')$.*

Proof: Let $y_{11} \equiv y(k_1, k_1')$, $y_{12} \equiv y(k_1, k_2')$, $y_{21} \equiv y(k_2, k_1')$, and $y_{22} \equiv y(k_2, k_2')$, define F_{ij} in the same way, and define $\Delta' \equiv (k_2' - k_1')/b$. We know from $y(k, k')$ being linear that $y_{12} - y_{11} = y_{22} - y_{21} = (k_2' - k_1')/b = \Delta'$. Now, we have

$$F_{22} - F_{21} = f(I - (y_{21} + \Delta')) + k_2^\alpha f(y_{21} + \Delta') - [f(I - y_{21}) + k_2^\alpha f(y_{21})]$$

$$= -[f(I - y_{21}) - f(I - (y_{21} + \Delta'))] + k_2^\alpha [f(y_{21} + \Delta') - f(y_{21})].$$

Similarly,

$$F_{12} - F_{11} = f(I - (y_{11} + \Delta')) + k_1^\alpha f(y_{11} + \Delta') - [f(I - y_{11}) + k_1^\alpha f(y_{11})]$$

$$= -[f(I - y_{11}) - f(I - (y_{11} + \Delta'))] + k_1^\alpha [f(y_{11} + \Delta') - f(y_{11})].$$

We will now compare the final expressions $F_{22} - F_{21}$ and $F_{12} - F_{11}$. Note that $y_{21} - y_{11} = -(1 - d)(k_2 - k_1)/b < 0$, which implies that $I - y_{21} > I - y_{11}$. Therefore, since f is strictly concave and $\Delta' > 0$, $f(I - y_{11}) - f(I - (y_{11} + \Delta'))$ is strictly greater than $f(I - y_{21}) - f(I - (y_{21} + \Delta'))$. It follows that $-[f(I - y_{21}) - f(I - (y_{21} + \Delta'))] > -[f(I - y_{11}) - f(I - (y_{11} + \Delta'))]$. That is, the first term of $F_{22} - F_{21}$ exceeds the first term of $F_{12} - F_{11}$. As for the second term, once more because $y_{21} < y_{11}$, and due to the strict concavity of f and $\Delta' > 0$, we have that $f(y_{21} + \Delta') - f(y_{21}) > f(y_{11} + \Delta') - f(y_{11})$. Finally, $k_2^\alpha > k_1^\alpha$ and f being strictly increasing implies that also the second term of $F_{22} - F_{21}$ exceeds that of $F_{12} - F_{11}$. QED.

Notice that this lemma would follow from F being globally concave in k and k' jointly. In our case, however, we know that it is not, unless $\alpha < 0.5$. Thus, the lemma is nontrivial, and relies on the functional forms adopted.

We now show that the policy function needs to be increasing, i.e., we have

Proposition: *If $k_2 > k_1$, then in any Markov-perfect equilibrium, we must have that $g(k_2) \geq g(k_1)$.*

Proof: Using the notation of the lemma, optimality implies

$$F(k_2, g(k_2)) + \beta\delta V(g(k_2)) \geq F(k_2, k') + \beta\delta V(k'))$$

for all feasible k'. Suppose now, by means of contradiction, that $g(k_1) > g(k_2)$. Presuming that choosing $g(k_1)$ at k_2 is feasible, the previous expression then yields

$$F(k_2, g(k_2)) + \beta\delta V(g(k_2)) \geq F(k_2, g(k_1)) + \beta\delta V(g(k_1))).$$

It follows that $F(k_2, g(k_2)) - F(k_2, g(k_1)) \geq \beta\delta[V(g(k_1))) - V(g(k_2))]$. Similarly, if choosing $g(k_2)$ is feasible at k_1, we must have that

$$F(k_1, g(k_1)) + \beta\delta V(g(k_1)) \geq F(k_1, g(k_2)) + \beta\delta V(g(k_2))),$$

so that $F(k_1, g(k_1)) - F(k_1, g(k_2)) \geq \beta\delta[V(g(k_2))) - V(g(k_1))]$. This inequality implies that $\beta\delta[V(g(k_1)))-V(g(k_2))] \geq F(k_1, g(k_2))-F(k_1, g(k_1))$. Now, adding the two inequalities we arrive at $F(k_2, g(k_2))-F(k_2, g(k_1)) > F(k_1, g(k_2))-F(k_1, g(k_1))$, which contradicts the lemma. Finally, we need to verify the feasibility assumed above. To see that $g(k_1)$ is feasible from k_2 and that $g(k_2)$ is feasible from k_1, note that what is feasible from k_i is the interval $(1-d)k_i + [0, Ib]$, for $i = 1, 2$. Thus, since $k_2 > k_1$ and $g(k_2) < g(k_1)$ by assumption, the required feasibility must follow. QED.

Notice that this proposition holds true independently of the value of β, so that it in particular applies to the standard model with time-consistent preferences. What is striking about this proposition is (i) that it holds despite F not being globally concave and (ii) whether or not preferences are time-inconsistent.

3.1 Continuous domain and differentiable decision rules

Assuming an interior solution, the first-order condition for this problem reads

$$(-u_1(x, ky) + ku_2(x, ky))y_2(k, k') + \beta\delta v'(k'),$$

and the envelope condition yields

$$v'(k) = (-u_1(x, ky)+ku_2(x, ky))(y_1(k, k')+y_2(k, k')g_k(k))+u_2(x, yk)y(k, k')+\delta v'(k')g_k(k) =$$

$$= (-u_1(x, ky)+ku_2(x, ky))(y_1(k, k')+g_k(k)y_2(k, k')(1- \ frac1\beta))+u_2(x, yk)y(k, k').$$

Thus, the final first-order condition reads (with arguments suppressed)

$$(-u_1 + ku_2)y_2 + \beta\delta\left((-u_1' + k'u_2')(y_1' + g_k'y_2'(1 - \frac{1}{\beta})) + u_2'y'\right) = 0.$$

We see that this condition collapses to that of the time-consistent case when $\beta = 1$. Compared to that case, there is another benefit—a new positive term in the expression—from saving more on the margin, assuming that culture consumption is increasing in k and that there is static overconsumption of culture. This reflects an added marginal value of consuming culture as it increases future culture accumulation, which is below what the present self would choose were he able to commit.

The interior steady state cannot be found in any easy manner. If the objective function is quadratic and $y(k, k')$ is linear, one can guess on a linear form for $g(k)$ and verify the guess, thus in particular delivering a steady state. However, if a closed-form solution is not available, which in general it is not, the determination of a steady state is fundamentally more complex than in the case where preferences are time-consistent, because the steady state depends on the value of g_k: the long-run level of k cannot be ascertained without determining the local dynamics around this level.

In an interior steady state, we have

$$(-u_1 + ku_2)(y_2(1 + \delta g_k(\beta - 1) + \beta\delta y_1) + \beta\delta u_2 y = 0.$$

Thus, compare models which share the value of $\beta\delta$ but where β and δ differ; suppose in one case $\beta = 1$ and in the other $\beta < 1$. Then, in the latter case, an additional term appears. The additional term is $(-u_1 + ku_2)y_2\delta g_k(\beta - 1)$: an additional marginal return to increasing k'. It is positive, since $-u_1 + ku_2 < 0$, $g_k > 0$, and $\beta < 1$. This indicates that the two models would have different steady states and that the one with a $\beta < 1$ would have a higher steady state: there is an additional motive for accumulation of culture capital; that is, the model with time inconsistency leads to higher cultural consumption.[9] This is not surprising: in the two models, the short-run discount rate is the same, and the long-run discount rate is higher in the case with $\beta < 1$, since the δ must be higher in that case; this is what explains the higher culture capital accumulation.

3.1.1 Numerical analysis

In the case of a continuous domain and our special functional-form assumptions, we have not found a closed-form solution for any value of $\alpha \in (0, 1)$. Therefore, in order to characterize equilibria, one would need to use numerical methods. One possible method for this is developed in Krusell, Kuruşçu, and Smith (2002). It could be used to look for a differentiable equilibrium function g satisfying the Euler equation locally. A second possibility would be to use "global methods". These rely on approximating the function g on a grid of values for k and interpolating in between grid points, either using cubic splines or some form of polynomial functions. The parameters of the cubic splines/polynomial functions would then be chosen so that the Euler equation holds on all grid points. Both these methods rely on the construction of g using the first-order condition. Thus, they do not verify sufficiency globally. This is a problem, especially in the present context of a potentially non-concave value function, due to the complementarity of present and future culture consumption. Moreover, it is known that in itself, time-inconsistency can lead to non-concave value functions. Thus, it would require new methods to search for a differentiable Markov-perfect equilibrium, which is beyond the scope of the current analysis. However, the purpose of introducing time-inconsistency of preferences here

[9] This conclusion will follow under concavity of $F(k, k')$, i.e., if $\alpha < 0.5$. In this case, it follows from our argument that the addition of a term on the left-hand side leads $\delta u_2 y$ to decrease. Since the expression $u_2 y$ is decreasing when $\alpha < 0.5$, this must imply that y must increase. If concavity is not met, the situation is more complicated.

is not primarily to find out how the smooth solution for g shifts in (k, k') space with parameter β. Rather, the purpose is to investigate whether equilibria of a different nature may exist. For this purpose, a continuous domain is not necessary.

There may also be non-differentiable equilibria in the context of the present model, when preferences are time-inconsistent. In particular, we conjecture that it is possible to construct stable steady states using step-function equilibria—equilibria where $g(k)$ is a step function, i.e., has a sequence of flat and vertical sections—following the work of Krusell and Smith (2003a). We will return to a discussion of such equilibria in the next section.

3.2 A discrete domain

We now look at a discrete domain, and we first focus on the very simplest case with only two possible values for culture capital. This case points to some difficulties that may arise in characterizing equilibria with time-inconsistent preferences. With a sufficiently "nice" structure, as in the case with a quadratic utility function and linear constraints, these problems can sometimes be avoided, but as we shall see in the case considered here, they do appear in other settings. After looking at the two-value setting, we consider setups with a large number of admissible values for culture capital and study equilibria using computational methods.

3.2.1 Two values for capital

With two values for capital, there are four possible values for the consumption of culture capital, y, which we will all consider to be feasible (i.e., the values of the parameters are such that all four values for y are between 0 and I), and hence, also four values for the other consumption good, x. Thus, we also have four values of flow utility that we denote F_{ij}, with i and j in $\{1, 2\}$. This case can be completely characterized: for every parameter configuration, it is possible to show what the equilibrium set will look like. However, here we will mainly show that the time inconsistency fundamentally changes the nature of outcomes, which will be illustrated with some particularly interesting cases.

Under time-consistent preferences, there is always a unique solution for the value function, v_1 and v_2, from the contraction mapping theorem, and given these values, optimal policies can be chosen without mixing.[10] Under time inconsistency, there is always at least one Markov-perfect equilibrium under mixed strategies; this follows from a standard fixed-point theorem.[11] However, (i) there may be no pure-strategy equilibrium and (ii) there may be multiple equilibria, with multiple (v_1, v_2) solving the equilibrium conditions.

We will mainly focus on pure-strategy equilibria. There are several possibilities. There may be either culture accumulation, i.e., the consumer may choose $k' = k_2 > k_1$, or not, i.e., $k' = k_1$. This gives four possible candidates, but one case cannot be optimal, namely, choosing $g(k_1) = k_2$ and $g(k_2) = k_1$, because this outcome violates the monotonicity of the decision rule: it violates the Proposition above, which was proved for a general case. Thus, there can be three kinds of equilibria: (i) choosing k_1 in both states; (ii) choosing k_2 in both states; and (iii) choosing to remain in whichever state is the starting point.

In an equilibrium of type (i), where k_1 is always chosen,

$$F_{11} + \beta\delta v_1 \geq F_{12} + \beta\delta v_2$$

and

$$F_{21} + \beta\delta v_1 \geq F_{22} + \beta\delta v_2,$$

with $v_1 = F_{11}/(1-\delta)$ and $v_2 = F_{21} + \delta F_{11}/(1-\delta)$. This implies that

$$F_{11} - F_{12} \geq \beta\delta(F_{21} - F_{11})$$

and

$$F_{21} - F_{22} \geq \beta\delta(F_{21} - F_{11}). \tag{3.4}$$

By the lemma, the former inequality is implied by (3.4). Thus, if inequality (3.4) is satisfied, an equilibrium of type (i) exists.

In an equilibrium of type (ii), where k_2 is always chosen,

$$F_{22} + \beta\delta v_2 \geq F_{21} + \beta\delta v_1$$

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[10] There may be indifference between policies, but probabilities may all be set to either 0 or 1.

[11] For example, Kakutani's fixed-point theorem can be used.

and

$$F_{12} + \beta\delta v_2 \geq F_{11} + \beta\delta v_1,$$

with $v_2 = F_{22}/(1-\delta)$ and $v_1 = F_{12} + \delta F_{22}/(1-\delta)$. This implies that

$$F_{22} - F_{21} \geq \beta\delta(F_{12} - F_{22})$$

and

$$F_{12} - F_{11} \geq \beta\delta(F_{12} - F_{22}). \tag{3.5}$$

By the lemma, the former inequality is implied by (3.5). Thus, if inequality (3.5) is satisfied, an equilibrium of type (ii) exists.

Finally, in an equilibrium of type (iii), where $g(k_1) = k_1$ and $g(k_2) = k_2$,

$$F_{11} + \beta\delta v_1 \geq F_{12} + \beta\delta v_2$$

and

$$F_{22} + \beta\delta v_2 \geq F_{21} + \beta\delta v_1,$$

with $v_1 = F_{11}/(1-\delta)$ and $v_2 = F_{22}/(1-\delta)$. This implies

$$F_{11} - F_{12} \geq \frac{\beta\delta}{1-\delta}(F_{22} - F_{11}), \tag{3.6}$$

and

$$F_{22} - F_{21} \geq \frac{\beta\delta}{1-\delta}(F_{11} - F_{22}). \tag{3.7}$$

Neither of these conditions in general imply the other. Thus, for an equilibrium of type (iii) to exist, both inequality (3.6) and inequality (3.7) must be verified.

Features of pure-strategy equilibria Now that the conditions for each of the pure-strategy equilibria to exist have been stated, a few remarks are in order. First, inequalities (3.4) and (3.5) cannot both be met at the same time: adding the two inequalities and rearranging gives:

$$[F_{21} + F_{12} - F_{11} - F_{22}](1 - \beta\delta) \geq 0.$$

The term in brackets, however, is strictly negative from the lemma, which thus rules out inequalities (3.4) and (3.5) from holding for the same set of parameter values. Thus, equilibria of type (i) and (ii) cannot coexist.

Second, and relatedly, both equilibria of type (i) and type (ii) have the feature that only short-run discounting is of importance, i.e., the value of $\beta\delta$: whether the conditions for existence of these equilibria are met only depends on the product of β and δ and not on these parameters separately. Intuitively, this is because in both these types of equilibria, the next period's choice of culture capital accumulation, k'', is independent of current actions. Therefore, how two adjacent periods in the future are compared (which is determined by δ) relative to how the next period is compared to the present period (which is determined by $\beta\delta$) is of no importance. For example, in state 2 of an equilibrium of type (i), the consumer must find it optimal not to deviate and choose little capital accumulation. Whether the deviation is optimal is only related to comparing $F_{21} + \beta\delta F_{11}$, which is what the equilibrium prescribes, with $F_{22} + \beta\delta F_{21}$, the deviation, because what the consumer then does is to choose k_2, regardless of the current actions. Equilibrium (iii) does not have this feature: whether it exists depends on β and δ separately.

Third, inequality (3.4) can be rearranged as

$$F_{22} - F_{21} \leq \frac{\beta\delta}{1-\beta\delta}(F_{11} - F_{22}).$$

Similarly, inequality (3.5) can be written as

$$F_{11} - F_{12} \leq \frac{\beta\delta}{1-\beta\delta}(F_{22} - F_{11}).$$

From these expressions, we realize that in the case of time-consistent preferences, when $\beta = 1$, it must be that inequality (3.4) is identical to the reverse of inequality (3.7) and that inequality (3.5) is the same as the reverse of inequality (3.6). This is an illustration of the point made earlier: under time-consistent preferences, one and only one of the three kinds of equilibria must exist (unless we are in a non-generic case where the inequalities are equalities, in which case the consumer is indifferent). That is, if type (i) exists, type (ii) cannot exist and (iii) cannot exist (except in the non-generic case where (3.4) holds with equality so that (iii) gives the same values as equilibrium (i)); if (ii) exists, (i) cannot exist and (iii) cannot exist (except ...); if (iii) exists, neither can (i) or (ii) (except ...), and it is not possible that none of the equilibria exist, because if equilibrium of type (iii) does not exist, then either equilibrium of type (i) has to exist or equilibrium of type (ii) must exist.

Welfare analysis Here, compute welfare effects of some different kinds of policies that might be imagined in this economy, and compare them to the equilibria. Two different kinds of welfare measures are available: that given by the vs and that given by the ws. For simplicity, do not explicitly consider policies, but rather look at whether a given equilibrium with an associated rule g can be improved upon, either in the sense of v or in the sense of w, for one or both of the states of nature. For example, consider parameter configurations such that the equilibrium is unique and is of type (i), i.e., always consume as little culture as possible: always choose state 1. Then the question would be whether a different decision rule, e.g., choose k_2 in state 2 and k_1 in state 1, might lead to higher welfare even though it is not an equilibrium. If that is the case, then any (unspecified) policy inducing people to follow this alternative would be desirable. Given that $v_1 = F_{11}/(1-\delta)$ and $v_2 = F_{21} + \delta F_{11}/(1-\delta)$ in an equilibrium of type i, and the corresponding future utilities in an equilibrium of type (iii) are $v_1 = F_{11}/(1-\delta)$ and $v_2 = F_{22}/(1-\delta)$, the equilibrium is worse than this specific alternative in state 2, and for the current self, if

$$F_{21} + \beta\delta\frac{F_{11}}{1-\delta} < F_{22} + \beta\delta\frac{F_{22}}{1-\delta}$$

and worse for the last-period self if

$$F_{21} + \delta\frac{F_{11}}{1-\delta} < F_{22} + \delta\frac{F_{22}}{1-\delta}.$$

Thus, one question would be whether one or both of these inequalities could be satisfied at the same time as inequality (3.4), which guarantees the existence of a type-(i) equilibrium, is satisfied. If so, it could be argued that the equilibrium is inefficient, and that the government should try to induce agents to choose state 2 instead of state 1, when they are in state 2.

Nonexistence of pure-strategy equilibria Let us now illustrate non-existence of pure-strategy equilibria under time-inconsistent preferences. First, take a case under time-inconsistent preferences where the primitives are such that an equilibrium of type (iii) exists, i.e., one with two steady states: where you start is where you end up. This implies that conditions (3.4) and (3.5), for equilibria of type (i) and (ii), respectively, are violated. Now select a new δ, which we call $\hat{\delta} > \delta$, and a $\beta < 1$, such that $\beta\hat{\delta} = \delta$. Thus, conditions (3.4) and (3.5) are still violated, so that neither equilibrium of type (i) nor type (ii) can exist. Furthermore, suppose, without loss of generality, that $F_{22} > F_{11}$. Then it is clear that while letting $\beta\hat{\delta} = \delta$, $\hat{\delta}$ can be increased close enough to 1 (and let β fall) that inequality (3.6) is violated, since $1/(1 - \hat{\delta})$ can be made to go to infinity.[12] Thus, with a judicious choice of discount factors, it appears that none of the equilibria exist.

Concerning the intuition for why a pure-strategy equilibrium does not exist here, note that what has happened due to the violation of condition (3.6) is that the consumer who is in state 1 now finds it worthwhile to deviate and "save more", i.e., increase culture consumption, and the reason for this is the time inconsistency produced by a very high δ: β is now significantly lower than 1. Thus, this consumer disagrees with his future selves, who are not very forward-looking, and he realizes that by switching from $k' = k_1$ to $k' = k_2$, he effectively ensures that k_2 will be chosen forever after: one always remains in this equilibrium.

It may appear counterintuitive that the equilibrium would not simply switch to one of type (ii) now, i.e., always choose k_2. However, we know that because $\beta\delta$ is the same as before, i.e., it is sufficiently low that it does not pay to choose k_2 in state 1, since that does not give the long-run benefit that it delivers in a type-(iii) equilibrium. So in this case, instead, there is a *mixed-strategy equilibrium* where the consumer randomizes between choosing k_1 and k_2 in state 1. In terms of comparative statics using δ, as it first reaches the level where condition (3.6) becomes an equality, the probability of choosing k_2 in state 1 is 0, but as it increases further, this probability is increased.[13] As it is increased, the benefit of choosing k_2

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[12] If $F_{22} < F_{11}$, use the same argument on inequality (3.7).

[13] The probability with which the agent randomizes between k_1 and k_2 must be chosen precisely so that (3.6) holds with equality. Denoting the probability of choosing k_1 by π, we have that $v_1 = \pi(F_{11} + \delta v_1) + (1 - \pi)(F_{12} + \delta v_2)$ and that $v_2 = F_{22}/(1 - \delta)$. This gives that $v_1 - v_2 =$

over k_1 is increased because of the accentuation of time-inconsistency—a discrepancy between the $\beta\delta$ which remains constant and the δ which is increasing—but there is a counteracting force making the consumer remain indifferent between k_1 and k_2: the long-run gain is now realized with lower and lower probability, since k_1 is less and less likely to be chosen in state 1.

Multiplicity To illustrate the multiplicity of equilibria, let us consider a case with some analytical convenience. So suppose first that we are in a world with time-consistent preferences satisfying the nongeneric case where (3.4) holds with equality and is identical to (3.7): $F_{21} - F_{22} = (\beta\delta/(1-\delta))(F_{22} - F_{11})$, with $\beta = 1$. Moreover, suppose that (3.6) is strictly satisfied (so that (3.5) is strictly violated). In words, this is the case where an equilibrium of type (i)—always keep culture accumulation low—"barely" exists and coexists with an equilibrium of type (iii), which also barely exists. In each case, the temptation to deviate is in state 2, where the consumer is indifferent between remaining in state 2—maintaining high culture consumption—and going to state 1; the equilibrium actions are thus different in (i) and (iii) in state 2—choose $k' = k_1$ in equilibrium (i) and choose $k' = k_2$ in equilibrium (iii)—but because of indifference, the associated values are the same. Suppose, in addition, that $F_{11} < F_{22}$.[14]

The formal idea is now to let β differ from 1 while maintaining $\beta\delta/(1-\delta)$, so that $F_{21} - F_{22} = (\beta\delta/(1-\delta))(F_{22} - F_{11})$ still holds; i.e., condition (3.7) still holds with equality: the type-(iii) case would thus still "barely" constitute an equilibrium. The expression $\beta\delta/(1-\delta)$ which is to be held constant, can be written as $[\beta\delta/(1-\beta\delta)][(1-\beta\delta)/(1-\delta)]$. Now, consider a change such that the first of these factors, $\beta\delta/(1-\beta\delta)$, decreases; this decrease would make the consumer not be indifferent in state 2 of the type-(i) equilibrium, but the consumer would rather strictly prefer to remain in state 2 over going to state 1. As a result, $(1-\beta\delta)/(1-\delta)$ would have to

$(\pi(F_{11} - F_{12}) + F_{12} - F_{22})/(1 - \pi\delta)$. This expression can be used together with equality of (3.6), which is stated as $F_{11} + \beta\delta v_1 = F_{12} + \beta\delta v_2$, to deliver an equation determining π. The solution is $\pi = (1/\delta - \beta(F_{22} - F_{12})/(F_{11} - F_{12}))/(1-\beta)$. Note here that this expression is still positive when δ has reached 1. The reason is that, by assumption, we have $F_{22} > F_{11}$ (the assumption upon which the non-existence of pure-strategy equilibria is based) and that $F_{11} + \beta\delta F_{12} > F_{12} + \beta\delta F_{22}$ (since, by assumption, the type-(ii) equilibrium does not exist), which together imply that $\beta(F_{22} - F_{12})/(F_{11} - F_{12}) < 1/\delta$.

[14] If it is not, a similar argument can be constructed.

increase, which requires β to be below 1 (and also makes δ increase).

How do these equilibria differ? If the economy starts in state 1, they do not differ: they both involve choosing to remain in state 1 forever. However, the two equilibria have very different long-run characteristics if the initial state is state 2. If equilibrium (i) is played, the consumer chooses to go to state 1 and then remain there forever, and if equilibrium (ii) is played, the consumer chooses to remain in state 2 forever.

What are the associated welfare levels? From the perspective of the current self, the two equilibria are identical. To see this, note that in the type-(i) case, the consumer obtains $w_2^i = F_{21} + (\beta\delta/(1-\delta))F_{11}$, whereas in the type-(iii) equilibrium, the resulting welfare is $w_2^{iiii} = F_{22} + (\beta\delta/(1-\delta))F_{22}$. But because of indifference in state 2 of the type-(iii) equilibrium, we see that w_2^i must equal w_2^{iii}. This feature is special and due to the way this equilibrium was constructed, there are other cases of multiplicity where the welfare of the current self differs across equilibria. Still, this case of equilibrium multiplicity is not without interest from a welfare perspective, because it could be argued that the more appropriate welfare measure is the one using δ as the discount factor, i.e., using the perspective of yesterday's self on choices today and in the future.[15] Thus, we have $v_2^i - v_2^{iii} = F_{21} - F_{22} + (\delta/(1-\delta))(F_{11} - F_{22})$, which gives more weight to the latter term, and since $F_{11} < F_{22}$, this implies that equilibrium of type (iii) is better. To summarize, in this example, welfare analysis suggests that the outcome with high long-run culture consumption is to be preferred over the one with low culture consumption, even though both these outcomes are equilibria.

3.2.2 Summary and implications

In this section, we summarize and draw some brief conclusions.

Multiplicity: optimism and pessimism in culture consumption The multiplicity of equilibria is a quite striking feature of a decision problem. An individual, left to his or her own devices, can solve the problem in different ways, some of which are strictly better than others, yet these different ways are all rational, in the sense

[15] See Chapter 1 for a justification and discussion.

of being equilibria to the multiple-selves game. Thus, two individuals in identical choice situations may choose to consume different amounts of culture and end up on different welfare levels. One individual chooses a great deal of culture, because the future selves of this person are expected to choose high levels of culture consumption, thereby making present culture consumption pay off, and the other chooses little culture because so will the future selves. Thus, we have yet another source of diversity in culture consumption: optimism and pessimism as multiple equilibria in the intrapersonal game.

Because of the potential for welfare-ranked equilibria, policy here seems particularly relevant to consider: governments could potentially help people not to have to fall into traps of pessimism! The idea is that a policy changes incentives and some policies may render equilibria unique and eliminate the pessimistic equilibrium. To further explore this idea, policy would need to be explicitly considered in the model; we leave this to future research.

Numerical methods for computing equilibria When preferences are not time-consistent, Markov-perfect equilibria can be expressed as functional equations but are not contraction mappings. This means that there is no guarantee that a Markov-perfect equilibrium will exist in the case of a continuous domain. For a discrete domain, these equilibria can be shown to exist if mixed strategies are allowed, but there is no guarantee that pure-strategy equilibria exist. When they do not, numerical computation is challenging; methods based on iteration on the functional equation defining equilibrium are then bound not to work. Suppose, for example, that there is an initial guess on a decision rule for the future selves, and that given this guess, we derive the future value functions, i.e., the vs. Then, the new policy rule is obtained by the maximization problem over k', given the vs. This problem will generically not deliver indifference between two alternatives. So if the equilibrium is unique and requires mixed strategies equilibrium, it will not be found with standard iteration.

3.2.3 Numerically computed equilibria

Here, we illustrate how more steady states—and more discontinuities in decision rules—can arise due to time inconsistency alone. This is done by numerically calculating pure-strategy equilibria, using a finite grid and forcing culture capital to lie on the grid. We will only show one example; a full characterization is beyond the scope of the present project.

Before displaying the example, we make two remarks. First, as shown in the previous section in a simple example with two grid points only, pure-strategy equilibria do not always exist. Indeed, in Krusell and Smith (2003b), a consumption-savings model with time-inconsistent preferences and a finite domain for capital is studied, and there pure-strategy equilibria did not tend to exist for fine grids. We believe that the same often holds true in this model. In particular, starting from a case with $\beta = 1$ and a stable steady state, we found that slight decreases in β led to nonconvergence when using pure strategies only. For much lower values of β, however, such that the whole decision rule would be expected to fall below the 45-degree line, we were able to find pure-strategy equilibria, and it is one of those that we will discuss below.

Second, and relatedly, we believe that at least in the case where $\alpha < 0.5$, step-function equilibria using mixed strategies, along the lines of the findings in Krusell and Smith (2003b) can be found. Indeed, we conjecture that step-function equilibria of the sort in Krusell and Smith (2003a) also exist for a continuous domain in this case, then as pure-strategy equilibria.

The example We start from a case with $\beta = 1$ and $\alpha > 0.5$**,** where the optimal law of motion for culture capital hugs the 45 degree line: there are two positive steady states. Lowering β to 0.83, we see a drastic change in the nature of the decision rule. Figure 3.11 shows the decision rules for these two values of β.[16]

There is a large number of jumps in the decision rule in the case of time-inconsistent preferences. Moreover, there are 7 positive steady states! Thus, if the initial level of culture capital is below the lowest positive steady state, the individual consumes less and less culture over time and converges to zero culture consumption.

[16] The parameter values are $\alpha = 0.53$, $I = 2.3$, $d = 0.75$, $b = 0.25$, and $\delta = 0.50$.

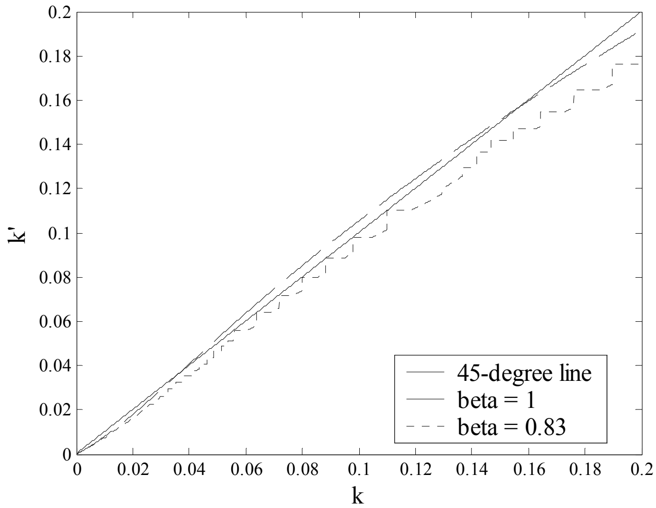

Figure 3.11:

If, however, the initial culture capital stock is higher than that, the long-run level of culture capital depends on exactly in what interval the initial level lies. In other words, we see a drastic increase in the long-run diversity of culture consumption due to time inconsistency in preferences.

To obtain intuition for the decision rule, let us focus on the typical segment between two steady states, denoted k_1 and k_3, stylized in Figure 3.12 below.

To understand how this behavior can be optimal over this range, first assume that over this range, there is a perceived lower bound on k'; we will later go back and argue why there is such a lower bound. If $k' \geq k_1$ is thus perceived as a constraint, this constraint will bind for a range of k from k_1 to the value denoted by k_2 in the graph. Over this range, the constraint binds strictly, and at k_2, it has ceased to bind strictly (the multiplier on the constraint at this point would be exactly zero). Thus, in the open interval (k_2, k_3), an increase in k increases k'. In the range $[k_1, k_2]$, all future selves, like the current self, will choose $k' = k_1$ and culture consumption will be $y = dk_1/b$ forever. In the range (k_2, k_3), all future selves will also choose $k' = k_1$ but culture consumption will be higher in the next period than $y = dk_1/b$ and only one period later fall to $y = dk_1/b$ and remain there forever. When k is as high as k_3, the individual chooses to remain at that level: an upward jump.

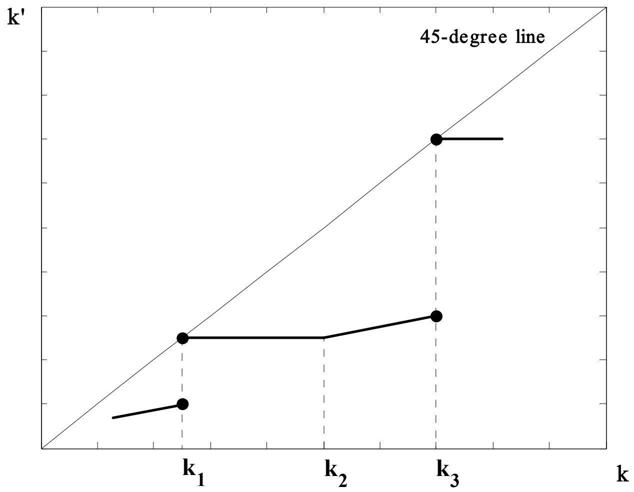

Figure 3.12:

Why is there an upward jump at k_3? Time-inconsistency implies that the current self would like the next self to be more forward-looking, and hence choose to consume more culture goods so as to accumulate culture capital. The current self thus sees a tradeoff between the low current culture accumulation, leading to $k'' = k_1$ (i.e., a starting level of k_1 two periods from now), which is consistent with the present-bias of the current self, and an effort which goes against the present-bias but ensures that the future selves end up at a much higher level of culture consumption. At k_3, the consumer is exactly indifferent between the low and the high level of culture consumption.

To explain the lower bound assumed, i.e., $k' \geq k_1$, suppose the current self to have a level of k slightly above k_1. For the same reason that the consumer is indifferent between remaining at k_3 and letting it fall drastically, the consumer at k_1 is indifferent between remaining at k_1, with high culture consumption, and significantly dropping culture consumption. For a consumer with a slightly higher initial value of k, thus, imagine that a choice of k' at a value slightly below k_1 were considered, thus moving beyond the artificially imposed constraint. The next self would then, by construction, be close to indifferent between keeping a high and a low level of culture consumption, but would marginally choose the lower level. What is the evaluation of this possibility of the current self? The current self significantly disagrees with the next self and places a larger weight on the future than does the next self. Therefore, it would *strictly* prefer the next self not to choose the lower value. This is why a choice of k' slightly below k_1 would give a jump down in utility and thus, we can locally view k_1 as a corner solution. Because of the strict preference, moreover, the constraint is strictly binding, which is the reason why there must be a flat section immediately to the right of k_1. Time inconsistency is thus essential in the argument, because it embodies the disagreement leading to flat sections and jumps.

In the stylized section, notice that over the range $[k_1, k_2)$, current culture consumption is strictly *decreasing* in k: more experience with culture leads to lower current culture consumption.

On a general level, the nonconvexity of the maximization problem leads to multiple local peaks in the objective functions, which will lead to decision rules with jumps. Because adjacent selves then disagree, flat sections are created, and so on.

4 Concluding remarks

We have explored a model where culture is viewed as a good involving taste cultivation. This model rather naturally implies that significant, endogenously generated, long- and short-run diversity in culture consumption follows, as long as culture is a relatively close substitute to the alternative good or activity. High substitutability seems quite natural, particularly since some forms of culture do not seem to be consumed at all, or almost not at all, by many consumers. However, one would like to estimate this degree of substitutability using a dynamic model of culture consumption like the present one; this is an important task for the future.

Though not considered here, a case where two culture goods compete can also be considered. Consider, for example, two distinct forms of "difficult-to-appreciate" music. Here, the assumption of high substitutability is a very natural one, and it seems clear that results similar to those obtained here would arise: the initial conditions would be of great importance, and a large diversity in taste generated by the dynamics of taste cultivation could be observed.

The possibility that preferences are time-inconsistent is particularly interesting when taste cultivation is important, because it suggests that consumers may benefit from government intervention. Here, it seems that government intervention may also have the effect of reducing the diversity of culture consumption, though we have only begun to analyze this issue. In particular, we did not explicitly introduce government policy here. Moreover, we did not provide a complete characterization of Markov-perfect equilibria under time-inconsistent preferences, but merely made some qualitative remarks using a simplified setup with a discretized domain for culture capital. Research that is left to future work thus includes a more complete exploration of equilibria in the present setup and similar setups as well as an explicit consideration of the role of policy.

Finally, one would also like to compare the results obtained here using the setup with multiple selves to that based on temptation and self-control developed by Gul and Pesendorfer (2001,2004). Multiplicity is less likely to result there, but interesting endogenous diversity due to the non-standard features of preferences may still be present.

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