Consumption-Savings Decisions with Quasi-Geometric Discounting: The Case with a Discrete Domain

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Abstract

How do individuals with time-inconsistent preferences—à la Strotz/Phelps/Pollak/Laibson—make consumption-savings decisions? We try to answer this question by considering the simplest possible form of consumption-savings problem, assuming that discounting is quasi-geometric. A solution to the decision problem is then a subgame-perfect equilibrium of a dynamic game between the individual's "successive selves". When the time horizon is infinite, we are left without a sharp answer: even when attention is restricted to Markov-perfect equilibria—using the agent's current wealth as state variable—we cannot rule out the possibility that two identical individuals in the exact same situation make different decisions! This paper deals with a discrete domain for capital, unlike Krusell and Smith (2003), which considers a continuous domain.

1 Introduction

The purpose of this paper is to study how an infinitely-lived, rational consumer with "quasi-geometric" discounting would make consumption and savings decisions.[1] We consider the idea that a consumer's evaluation of "utils" at different points in time does not have take the form of an aggregate with geometric weights. This idea was suggested first by Strotz (1956), and later elaborated on by Pollak (1968), Phelps and Pollak (1968), Laibson (1994, 1997) and others. Quasi-geometric discounting leads to time-inconsistent preferences: the consumer changes his mind over time regarding the relative values of different consumption paths. One version of this inconsistency takes the form of extreme short-term impatience. That formulation seems attractive based on introspection. The recent literature also emphasizes behavioral studies (such as Ainslie (1992)) as a motivation for a departure from geometric discounting. This literature documents "preference reversals", and it generally argues that time-inconsistency is as ubiquitous as risk aversion. This information is too important to dismiss: at the very least, there is no definite argument against a departure from geometric discounting, and since models with time-inconsistency potentially can have very different positive and normative properties than standard models, they deserve to be studied in more detail. That is what we set out to do here.

We assume that time is discrete, and that the consumer cannot commit to future actions. We interpret rationality as the consumer's ability to correctly forecast his future actions: a solution to the decision problem is required to take the form of a subgame-perfect equilibrium of a game where the players are the consumer and his future selves. We restrict attention to equilibria which are stationary: they are recursive, and Markov in current wealth; that is, current actions cannot depend either on time or on any other history than that summarized by current wealth.

The consumption/savings problem is of the simplest possible kind: there is no uncertainty, and current resources simply have to be divided into current consumption and savings. Utility is time-additive with quasi-geometric discounting, and the period utility function is strictly concave. We assume that the consumer operates a technology which has (weakly) decreasing returns in its input, capital (that is, savings from last period). A special case is

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[1]We mean by the term quasi-geometric a sequence which is geometric from the second date and on. The term "quasi-hyperbolic" has been used in the literature with the same meaning—see, e.g., Laibson (1997). Laibson's use, presumably, is motivated by trying to mimic approximately a true (generalized) hyperbolic function, which is possible within a subset of the quasi-geometric class. Mathematically, however, quasi-geometric is clearly a more appropriate term, and since we are interested in this entire class, as opposed to the subset mimicking the hyperbolic case, we opt for this term.

that of an affine production function, where the return is constant; this special case can be interpreted as one with a price-taking consumer who has a constant stream of labor income and can save at an exogenous interest rate. We do not study interaction between consumers in this paper.

Our main finding is one of indeterminacy of equilibria. That is, the restriction to Markov equilibria does not reduce the set of equilibria to a small number. First, there is indeterminacy in terms of long-run outcomes of the consumption/savings process: there is a continuum of stationary points to which the consumer's capital holdings may converge over time. Second, associated with each stationary point is a continuum of equilibria. Put simply, our theorizing does not allow us to rule out the possibility that two identical consumers placed in the same environment make radically different decisions, both in the short and the long run.

What is the origin of the indeterminacy? Almost by definition, one important component is expectations: equilibria can be thought of as "expectations-driven". Optimism and pessimism regarding your own future behavior is a real phenomenon in our model. The expectations concern future savings behavior. In the time-consistent model, the expectations of future savings behavior are not relevant, since there is agreement on that behavior: an envelope theorem applies. If, instead, the consumer places a higher relative weight on consumption two periods from now than does his next-period self, then a high savings propensity of his next self is an added bonus from saving today. Therefore, what he believes about this future savings propensity is important. One consumer may decide to save a lot because he expects himself to save a lot in the future, thereby giving a high return to saving today; another instead expects to consume a lot next period, thus lowering the incentives to save now. Another important component in our equilibrium construction is a discontinuous policy rule for savings. That is, we employ locally extreme savings propensities to make the construction alluded to above. The precise intuition for our indeterminacy is developed in Krusell and Smith (2003); in this note, we merely study the discrete-domain case. Related work on this topic can be found in Asheim (1997) and Vieille and Weibull (2003, 2008).

The indeterminacy that we document in this paper has not been noted in the existing literature on consumption-savings decisions with quasi-geometric discounting. Laibson (1994) and Bernheim, Ray, and Yeltekin (1999) find indeterminacy in settings similar to the one studied in this paper, but they rely on history-dependent ("trigger") strategies. In this paper, instead, we restrict ourselves to Markov equilibria in which current consumption decisions depend only on current wealth.

Harris and Laibson (2000) study a consumption-savings problem in which the agent faces a constant interest rate and stochastic labor income. Their framework is closely related to ours, which allows an affine production function as a special case. The difference is that we consider a deterministic environment; their analysis does not contain ours as a special case, and it seems important to understand the deterministic case separately. In addition, we are able to provide an explicit characterization of equilibria near a stationary point; Harris and Laibson provide existence, but not uniqueness nor explicit solutions.

We lay out our basic framework, using recursive methods, in Section 2. That model allows capital to be any number on an interval of the real line. To illustrate the setup, we parameterize the model—logarithmic utility and Cobb-Douglas production—and derive an analytical solution for this case. In Section 3, we then restrict the domain for capital to a finite grid. The discrete-domain case allows us to demonstrate and discuss our multiplicity results in a concrete and simple way. We also use it to study whether there are simple domination arguments to rule out all equilibria but one. We therefore spend some time analyzing the simplest possible consumption-savings problem: capital can take on only two values, high and low. Finally, we use the discrete-domain case as a way of computing equilibria numerically.

2 The setup

2.1 Primitives

Time is discrete and infinite and begins at time 0.[2] There is no uncertainty. An infinitely-lived consumer derives utility from a stream of consumption at different dates. We assume that the preferences of the individual at time t are time-additive, and that they take the form

$$U_t = u_t + \beta_1 u_{t+1} + \beta_2 u_{t+2} + \beta_3 u_{t+3} + \dots.$$

The variable u_t denotes the number of utils at time t; it is implicit that these utils are derived from a function $u(c_t)$, where c_t is consumption at time t. This formulation thus embodies an assumption of stationarity: the discounting at any point in time has the form $1, \beta_1, \beta_2, \dots$. The same consumer at $t+1$ thus evaluates utility as follows:

$$U_{t+1} = u_{t+1} + \beta_1 u_{t+2} + \beta_2 u_{t+3} + \beta_3 u_{t+4} + \dots.$$

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[2]Barro (1997) studies a continuous-time model without uncertainty where the consumer's discounting is not exponential.

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Clearly, the lifetime utility evaluations at t and $t+1$ express different views on consumption at different dates, unless $\beta_{t+k+1}/\beta_{t+k}$ is the same for all t and k and equal to β_1, that is, unless discounting is geometric: $\beta_t = \beta^t$ for some β. We take the view here that geometric discounting is a very special case and that the a priori grounds to restrict attention to it are weak. We consider a very simple departure from geometric discounting: quasi-geometric discounting. Quasi-geometric discounting can be expressed with two parameters, β and δ. The weights on future utils are $1, \beta\delta, \beta\delta^2, \beta\delta^3, \ldots$. That is, discounting is geometric across all dates excluding the current date:

$$U_t = u_t + \beta\left(\delta u_{t+1} + \delta^2 u_{t+2} + \delta^3 u_{t+3} + \ldots\right).$$

The case where $\beta < 1$ corresponds to particular short-run impatience ("I will save, just not right now"), and $\beta > 1$ represents particular short-run patience ("I will consume, just not right now"). The case $\beta = 1$, of course, is the standard, time-consistent case.

It is straightforward to generalize quasi-geometric discounting: the weights would then be general for T periods, and geometric thereafter. Pure hyperbolic discounting corresponds to the case $\beta_t = 1/(t+1)$, which we do not consider here. In most of our analysis, we will restrict attention to $\delta < 1$, since our resources are bounded. With growing resources, it is possible to allow a δ larger than 1 if the utility function takes a certain form.

We assume that the period utility function $u(c)$ is strictly increasing, strictly concave, and twice continuously differentiable. The consumer's resource constraint reads

$$c + k' = f(k)$$

where k is current capital holdings, f is strictly increasing, concave and twice continuously differentiable. We will focus on the case where f is strictly concave, but this assumption is not essential for our main results.

2.2 Behavior: modelling choices

How do we model the decision making? We use four principles:

1. We assume that the consumer cannot commit to future actions.

2. We assume that the consumer realizes that his preferences will change and makes the current decision taking this into account.

3. We model the decision-making process as a dynamic game, with the agent's current and future selves as players.

4. We focus on (first-order) Markov equilibria: at a moment in time, no histories are assumed to matter for outcomes beyond what is summarized in the current stock of capital held by the agent.

Some comments are in order. The first of the principles makes the problem different than the standard case. With commitment, decisions could be analyzed starting at time 0 in an entirely standard fashion (using recursive methods) and only the decisions across time 0 and the rest of time would be different. That decision would be straightforward given an indirect utility function representing utility at times 1, 2 and on. Moreover, commitment is not an unrealistic assumption. Notice that commitment to consumption behavior in practice would require a demanding monitoring technology and might be quite costly. 401(k) plans do not provide commitment to consumption, unless there are other restrictions, such as borrowing constraints. We do not consider such constraints here. One could consider how access to a costly monitoring technology would alter the analysis. We leave such an analysis as well to future work. Of course, the ability to overcome the commitment problem may be a crucial ability of a consumer, and it deserves to be studied more.

Our second principle is what we interpret rationality to mean in this framework. We would not want to abandon it: systematic prediction errors of one's own future behavior are not studied in the time-consistent model, and we do not want to study them here. Moreover, studying such prediction errors does not require time-inconsistent preferences.

Our third principle is the same as that suggested and adopted in the early literature on time-inconsistent preferences. Our fourth principle is more of a restriction than a principle: we do not study history-dependent equilibria with the hope of arriving at sharper predictions. There is perhaps also a sense in which we think bygones should be bygones on the level of decision-making. There is also existing work where bygones are not bygones: Laibson (1994) and Bernheim, Ray, and Yeltekin (1999) study similar models and allow history dependence. The set of equilibria can certainly be expanded in this way.

2.3 A recursive formulation

Assume that the agent perceives future savings decisions to be given by a function $g(k)$:

$$k_t = g(k_{t-1}).$$

Note that g is time-independent and only has current capital as an argument.

The agent solves the "first-stage" problem

$$W(k) \equiv \max_{k'} u(f(k) - k') + \beta \delta V(k'),$$

where V is the indirect utility of capital from next period on. In turn, V has to satisfy the "second-stage" functional equation

$$V(k) = u(f(k) - g(k)) + \delta V(g(k)).$$

Notice that successive substitution of V into the agent's objective generates the right objective if the expectations of future behavior are given by the function g.

A solution to the agent's problem is denoted $\tilde{g}(k)$. We have an equilibrium if the fixed-point condition $\tilde{g}(k) = g(k)$ is satisfied for all k.

The fixed-point problem in g cannot be expressed as a contraction mapping. For a given (bounded and continuous) g, it is possible to express the functional equation in V as a contraction mapping. However, continuity of g does not guarantee that V is concave, and it is not clear that the maximization over k' problem has a unique solution. This also implies that $\tilde{g}$ may be discontinuous.

A simple parametric example can be used as an illustration of the recursion. Suppose $u(c) = \log(c)$ and $f(k) = Ak^\alpha$, with $\alpha < 1$. Then it is straightforward to use guess-and-verify methods to solve for the following solution:

$$k' = \frac{\alpha\beta\delta}{1 - \alpha\delta(1 - \beta)} Ak^\alpha$$

and

$$V(k) = a + b \log k$$

with steady state

$$k_{ss} = \left(\frac{\alpha\beta\delta A}{1 - \alpha\delta(1 - \beta)} \right)^{\frac{1}{1-\alpha}}.$$

This solution gives a lower steady state than with $\beta = 1$.[3]

It is easy to check that, for this example, the time-consistent behavior thus solved for actually coincides, in the first period, with the behavior that would result in the commitment solution.

An algorithm for numerical computation of equilibria is suggested directly from our recursive problem: pick an arbitrary initial V, solve for optimal savings and obtain a decision rule, update V, and so on. This algorithm is similar to value function iteration for the standard time-consistent problem. If the initial V is set to zero, it is also equivalent to how a finite-horizon problem would normally be solved. It turns out that this algorithm does not

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[3]The coefficients a and b are given by $b = \alpha/(1 - \alpha\delta)$ and $a = (\log(A - d) + \delta b \log(d))/(1 - \delta)$, where $d = \beta\delta bA/(1 + \beta\delta b)$.

work here. Typically, it leads to cycling.[4] Similarly, an algorithm that starts with a guess on g, solves for V from the second stage condition, and then updates g (say, by a linear combination between g and $\tilde{g}$) also does not work: it produces cycles. These two algorithms produce cycles even when u and f are of the parametric form we discussed above—when g is known to be log-linear—and even an initial condition very close to the exact solution is given. As we will see, the analysis in the following sections suggests a reason for the apparent instability of these algorithms: there are other solutions to the fixed-point problem in g that are not continuous, and the function approximations we used in the above algorithms rely on continuity (for example, we use cubic splines).

We now turn to a version of our model with a discrete state space.

3   The case of a discrete domain

We now assume that capital can only take a finite number of values: $k \in \{k_1, k_2, \ldots, k_I\}$. We make the following assumptions:

1. *Consumption-savings:* $u_{21} > u_{11} > u_{12}$, $u_{21} > u_{22} > u_{12}$.

2. *Strict concavity of u:*
$$u_{ij} - u_{ik} > u_{i'j} - u_{i'k}.$$
   for $i < i'$ and $j < k$.

3. *Impatience:* $\beta < 1$ and $\delta < 1$.

Define $\pi_{ij} \in [0, 1]$ to be the probability of going from state i to state j. Given π (a set of π_{ij}'s), find the value function given uniquely by the V_i's solving the contraction

$$V_i = \sum_j \pi_{ij} \left(u_{ij} + \delta V_j \right)$$

for all i (this is a linear equation system). This gives $V(\pi)$. The fixed-point condition requires

$$\pi_{ij} > 0 \Rightarrow j \in \arg\max_k \; [u_{ik} + \beta\delta V_k(\pi)].$$

Proposition 1: *There exists a mixed-strategy equilibrium for the economy with discrete domain.*

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[4]The algorithm may converge if g is approximated with very low accuracy (with few grid points, or with an inflexible functional form).

Proof: This is shown with a straightforward application of Kakutani's fixed-point theorem.
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It is also possible to show monotonicity of the decision rules:

Proposition 2: *The decision rule is monotone increasing, that is, if positive probability is put on k at i and i′ is larger than i, then the choice at i′ cannot have positive probability on j < k.*

Proof: Suppose not.

$$u_{ik} + \beta\delta V_k \geq u_{ij} + \beta\delta V_j$$

and

$$u_{i'j} + \beta\delta V_j \geq u_{i'k} + \beta\delta V_k$$

can be combined into

$$u_{ij} - u_{ik} \leq u_{i'j} - u_{i'k}$$

which violates strict concavity. ∎

Monotonicity is a very useful property for understanding the behavior of the consumer in this model. It will be used repeatedly below. The proof of monotonicity does not use discreteness, and therefore the monotonicity property also applies when the domain for capital is continuous.

3.1 The 2-state case

We study the simplest possible case in some detail: the case where capital can take only two values, 1 and 2 ($k_1 < k_2$). We will use the short-hand ij for an equilibrium where the decision in state 1 is to go to state i and the decision in state 2 is to go to state j, $i, j \in \{1, 2\}$; further, $i\pi$ refers to an equilibrium where there is mixing in state 2 (for some specific probability), and πj refers to mixing in state 1.

The characterization of the set of equilibria is that the parameter space (β, δ, and the u_{ij}'s) breaks into 6 regions:

Proposition 3: *Generically, there are six possible equilibrium configurations; each of the following characterizes a region:*

    1. A unique "no-saving" equilibrium: 1 → 1 and 2 → 1.

    2. A unique "saving" equilibrium: 1 → 2 and 2 → 2.

    3. A unique "status-quo" equilibrium: 1 → 1 and 2 → 2.

    4. No pure-strategy equilibrium: 1 → π and 2 → 2 (long-run saving).

5. *Three equilibria:*

  (a) *1 → 1 and 2 → 1 (no-saving).*

  (b) *1 → 2 and 2 → 2 (saving).*

  (c) *1 → 1 and 2 → π (no-saving).*

6. *Three equilibria:*

  (a) *1 → 1 and 2 → 1 (no-saving).*

  (b) *1 → 1 and 2 → π (no-saving).*

  (c) *1 → π and 2 → 2 (saving).*

Proof: See Appendix 1. ∎

Notice that regions 1–3 are expected and standard; region 4 is a case where no pure strategy equilibrium exists; and the remaining two cases have multiplicity. We will discuss their interpretation below. As shown in the proof of Proposition 3, regions 4–6 disappear for $\beta = 1$.

When there is more than one equilibrium, there are three. Two of these are very different in character: they lead to different long-run outcomes. The third is a mixed version of one of the others, with the same long-run outcome (equilibrium 5c is very similar to 5a, and 6b to 6a). The essential character of each of the two equilibria is: if your future self is a saver, so are you; if not, then neither are you.

The idea that there are multiple solutions to a decision problem is conceptually disturbing: faced in a given situation, what will the consumer do? Our theory does not provide an answer, or, it says several things can happen. Identical consumers, apparently, can make different decisions, rationally, in the same situation.

Can we interpret this as there being room for "optimism" and "pessimism" to influence decisions? These terms should, if used appropriately, refer to utility outcomes, about which we have remained silent so far. The fact is that, in our 2-state economy, equilibria with long-run savings are better than those without: they give higher current life-time utility, independently of the starting condition, than no-saving equilibria.[5] In this sense, there is a free lunch here: just be optimistic, it is not associated with costs!

Of course, the free lunch aspect suggests a natural refinement of equilibria, one which has a renegotiation character: why stick to expectations which can be replaced with better ones? This refinement seems to work well in this application. However, it turns out that this refinement is problematic when there are more than two possible states for capital. The reason is that, in general, there are parameter regions (which become large when the number

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[5]In fact, all equilibria are ranked in this sense.

of states becomes large) where a utility ranking across equilibria does not exist. For example, state i might give equilibrium A higher utility than equilibrium B, whereas in state j the reverse is true; moreover, state i might lead to state j under equilibrium A. That is, if one picks equilibrium A now, one will want to change one's mind later. That means that this refinement is not time-consistent, and therefore not useful.[6]

This means, as far as we can tell, absent other useful refinement concepts, that there might be room for optimism and pessimism. Of course, these terms now have a more restricted meaning, since an equilibrium with optimism today (in terms of current utility) may imply pessimism in the future, and vice versa.

3.2 More than 2 states

With more than two states, we resort to numerical methods for finding equilibria. To find equilibria, we either perform exhaustive search (which of course is a slow method, prohibitively so except for a very small number of states) or iterate on a fixed-point mapping from randomly selected initial conditions for π (this algorithm is fast, but will miss some equilibria, at least those which are "unstable").[7]

Several questions are relevant here:

- As the grid becomes finer, will the multiplicity expand, remain unchanged, or shrink?

- As the grid becomes extremely fine, and there is some hope that the solution approximates a continuous state-space solution, what are the properties of such a solution?

- If we restrict parameters to replicate log/Cobb-Douglas assumptions as closely as possible, will the analytical solution be found? Will other solutions continue to exist?

In general, the findings are: the multiplicity does not go away as the grid becomes finer, equilibria are not ranked in general, there is always some mixing when the grid is fine enough, the decision rules look "funny", and the analytical solution to the log/Cobb-Douglas case is not one of the equilibria that is produced by the algorithm.

We will illustrate the equilibrium features in Figures 1, 2, and 3. They depict the policy rule for capital, given current capital, and are constructed based on the case of 150 grid

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[6]Asheim (1997) proposes a concept called revision-proofness as a refinement to subgame-perfect equilibria and applies it in a context of time-inconsistent decision-making. He shows a specific example, similar to the present one and featuring a discrete state space, where a revision-proof equilibrium exists. Here, one would exist in the 2-state case, but not in general in the multi-state case.

[7]In order to find a mixed-strategy equilibrium, the latter method iterates until near indifference, makes a specific guess of indifference and solves for the equilibrium given this guess, and finally checks all equilibrium conditions.

points. The parameters are chosen based on the log/Cobb-Douglas specification; the analytical decision rule and the 45-degree line are also graphed in each of the figures. We found 30 equilibria in this case. If these are all the stable equilibria, there should be an odd number in addition.

The general features are as follows: decision rules seem smooth over some intervals, but have jumps. For a given decision rule, there is always a single stationary point. The stationary point is reached, from the right, by a flat section, and from the left, by a "creeping up along the 45-degree line". The "creeping up" actually occurs with mixing: these points are mixing the 45-degree line with a grid point above it. Mixing does not occur anywhere else. We will draw heavily on these features when we construct equilibria in the case of a continuous state space in the next section.

Comparing stationary points to the analytical case, Figure 1 has its stationary point above, Figure 3 below, and Figure 2 at the stationary point of the analytical solution. The equilibrium in Figure 1 actually dominates the other equilibria in the figures, but there is another equilibrium with which it cannot be ranked.

3.3   Additional comments

When the discrete-state model is solved backwards, that is, when a finite-horizon version of the model is solved, there is, as expected, a unique equilibrium for every time horizon. As the time horizon goes to infinity, there is sometimes no convergence in policy rules and value functions: a cycle is reached. "Sometimes" is always when there are many grid points. Intuitively, then, all equilibria we find with our other computational method have mixing, and mixing equilibria will not be found with backward-solving: they will not exist, generically, with a finite horizon. In the two-state case, for example, in region 4, where there is no pure-strategy equilibrium, there is lack of convergence. In regions 1–3, there is convergence to the unique pure-strategy equilibrium, and in regions 5 and 6 there is convergence to equilibrium (a): the no-saving equilibrium. Thus, the saving equilibrium seems to require an infinite horizon to be implementable.

4   Remarks

In this paper, we study the consumption-saving decisions of a consumer who has time-inconsistent preferences in the form of a departure from geometric discounting. Our analysis includes as a special case the simplest possible consumption-savings problem in which

a price-taking consumer faces a constant exogenous interest rate and receives a constant stream of labor income. We make no restrictions on the period utility function save for concavity. When the time horizon is infinite, we find that the dynamic game played between the consumer's successive selves is characterized by a severe multiplicity of equilibria. This multiplicity arises even though we restrict attention to Markov equilibria. We have explored versions of this setting with random shocks to productivity. In such a setting, there is still multiplicity.

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Appendix

This appendix contains the proof of Proposition 3 in Section 3.1.

First, conditions for each type of equilibrium to exist are given below (it is implicit that the consumption-savings and strict concavity assumptions are required in addition to the stated conditions). After these conditions are given, the proof is provided.

- $1 \to 1$ and $2 \to 1$. We have, normalizing so that $u_{11} \equiv 1$,

$$v_1 = 1 + \delta v_1$$

  and

$$v_2 = u_{21} + \delta v_1$$

  which implies

$$v_1 = \frac{1}{1 - \delta}$$

  and

$$v_2 = u_{21} + \frac{\delta}{1 - \delta}.$$

  This equilibrium exists if

$$1 + \beta \delta \frac{1}{1 - \delta} \geq u_{12} + \beta \delta (u_{21} + \frac{\delta}{1 - \delta})$$

  and

$$u_{21} + \beta \delta \frac{1}{1 - \delta} \geq u_{22} + \beta \delta (u_{21} + \frac{\delta}{1 - \delta}).$$

  These expressions simplify to

$$1 - u_{12} \geq \beta \delta (u_{21} - 1) \quad \text{and} \quad u_{21} - u_{22} \geq \beta \delta (u_{21} - 1).$$

  The latter of these implies the former, given the concavity assumption. Therefore this type of equilibrium exists if the latter is met.

- $1 \to 2$ and $2 \to 1$. This equilibrium cannot exist since it violates monotonicity.

- $1 \to 1$ and $2 \to 2$. We have

$$v_1 = 1 + \delta v_1$$

  and

$$v_2 = u_{22} + \delta v_2$$

  which implies

$$v_1 = \frac{1}{1 - \delta}$$

  and

$$v_2 = \frac{u_{22}}{1 - \delta}.$$

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This equilibrium exists if

$$1 + \beta\delta\frac{1}{1-\delta} \geq u_{12} + \beta\delta\frac{u_{22}}{1-\delta}$$

and

$$u_{22} + \beta\delta\frac{u_{22}}{1-\delta} \geq u_{21} + \beta\delta\frac{1}{1-\delta}.$$

These expressions simplify to

$$1 - u_{12} \geq \beta\delta\frac{u_{22} - 1}{1-\delta} \quad \text{and} \quad u_{21} - u_{22} \leq \beta\delta\frac{u_{22} - 1}{1-\delta}.$$

- $\underline{1 \rightarrow 2 \text{ and } 2 \rightarrow 2.}$ We have

$$v_1 = u_{12} + \delta v_2$$

and

$$v_2 = u_{22} + \delta v_2$$

which implies

$$v_1 = u_{12} + \frac{\delta}{1-\delta}u_{22}$$

and

$$v_2 = \frac{u_{22}}{1-\delta}.$$

This equilibrium exists if

$$u_{12} + \beta\delta\frac{u_{22}}{1-\delta} \geq 1 + \beta\delta\left(u_{12} + \frac{\delta}{1-\delta}u_{22}\right)$$

and

$$u_{22} + \beta\delta\frac{u_{22}}{1-\delta} \geq u_{21} + \beta\delta\left(u_{12} + \frac{\delta}{1-\delta}u_{22}\right).$$

These expressions simplify to

$$1 - u_{12} \leq \beta\delta(u_{22} - u_{12}) \quad \text{and} \quad u_{21} - u_{22} \leq \beta\delta(u_{22} - u_{12}).$$

The former of these implies the latter, given the concavity assumption. Therefore this type of equilibrium exists if the former is met.

- $\underline{1 \rightarrow \pi \text{ and } 2 \rightarrow 1.}$ This equilibrium cannot exist since it violates monotonicity.

- $\underline{1 \rightarrow \pi \text{ and } 2 \rightarrow 2.}$ We have

$$1 + \beta\delta v_1 = u_{12} + \beta\delta v_2$$

and

$$v_2 = u_{22} + \delta v_2$$

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which implies
$$v_1 = \frac{u_{22}}{1-\delta} + \frac{u_{12}-1}{\beta\delta}$$

and
$$v_2 = \frac{u_{22}}{1-\delta}.$$

The mixing probability satisfies
$$v_1 = \pi(1+\delta v_1) + (1-\pi)(u_{12}+\delta v_2),$$

implying
$$\pi = \frac{1}{1-\beta}\left(\frac{1}{\delta} + \beta\frac{u_{12}-u_{22}}{1-u_{12}}\right).$$

This equilibrium exists if
$$u_{22} + \beta\delta\frac{u_{22}}{1-\delta} \geq u_{21} + \beta\delta\left(\frac{u_{22}}{1-\delta} + \frac{u_{12}-1}{\beta\delta}\right),$$

which is unrestrictive since it is equivalent to concavity, $\pi \geq 0$, that is,
$$1 - u_{12} \geq \beta\delta(u_{22}-u_{12})$$

and $\pi \leq 1$, that is,
$$1 - u_{12} \leq \frac{\beta\delta(u_{22}-u_{12})}{1-\delta(1-\beta)}.$$

- <u>$1 \to 1$ and $2 \to \pi$.</u> We have
$$v_1 = 1 + \delta v_1$$

and
$$u_{21} + \beta\delta v_1 = u_{22} + \beta\delta v_2$$

which implies
$$v_1 = \frac{1}{1-\delta}$$

and
$$v_2 = \frac{1}{1-\delta} + \frac{u_{21}-u_{22}}{\beta\delta}.$$

The mixing probability satisfies
$$v_2 = \pi(u_{21}+\delta v_1) + (1-\pi)(u_{22}+\delta v_2),$$

implying
$$1 - \pi = \frac{1}{1-\beta}\left(\frac{1}{\delta} + \beta\frac{1-u_{21}}{u_{21}-u_{22}}\right).$$

This equilibrium exists if
$$1 + \beta\delta v_1 \geq u_{21} + \beta\delta v_2$$

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which is automatically met since it is equivalent to concavity, and $1 - \pi \geq 0$, that is,

$$u_{21} - u_{22} \geq \beta\delta(u_{21} - 1)$$

and $1 - \pi \leq 1$, that is,

$$u_{21} - u_{22} \leq \frac{\beta\delta(u_{21} - 1)}{1 - \delta(1 - \beta)}.$$

- $\underline{1 \rightarrow 2 \text{ and } 2 \rightarrow \pi.}$ This equilibrium cannot exist since it violates monotonicity.

- $\underline{1 \rightarrow \pi \text{ and } 2 \rightarrow \pi.}$ This equilibrium also cannot exist since it violates monotonicity.

In each of the cases when conditions for existence are given it is straightforward to see that parameter values do exist such that the given conditions are met. We now turn to discussing the possible coexistence of equilibria for given parameter values. The possible equilibria are denoted 11, 12, 22, π2, and 1π (referring to the decision in states 1 and 2, respectively). We assume in this section that β and δ are less than 1.

We now prove the proposition by methodically going through all possibilities. First we prove six facts.

- $\underline{22 \text{ does not coexist with any other equilibrium.}}$ 22 requires $1 - u_{12} \leq \beta\delta(u_{22} - u_{12})$. Let us consider each alternative equilibrium in turn.

  The condition for 11 is $u_{21} - u_{22} \geq \beta\delta(u_{21} - 1)$. Combining it with the condition for 22 we obtain

  $$1 - u_{12} - u_{21} + u_{22} \leq \beta\delta(u_{22} - u_{12} - u_{21} + 1)$$

  which is a contradiction given strict concavity and $\beta\delta < 1$.

  One of the conditions for the 12 equilibrium is that $(1 - \delta)(1 - u_{12}) + \beta\delta \geq \beta\delta u_{22}$. The condition for 22 is $\beta\delta u_{22} \geq 1 - u_{12} + \beta\delta u_{12}$. But these are inconsistent since $(1 - \delta)(1 - u_{12}) + \beta\delta - (1 - u_{12} + \beta\delta u_{12}) = -\delta(1 - \beta)(1 - u_{12}) < 0$.

  The π2 equilibrium violates the 22 condition immediately if $\pi > 0$; if $\pi = 0$ it reduces to the 22 equilibrium.

  The 1π equilibrium, finally, requires $u_{21} - u_{22} \geq \beta\delta(u_{21} - 1)$, or $u_{22} \leq (1 - \beta\delta)u_{21} + \beta\delta$ which is strictly less than $(1 - \beta\delta)(1 - u_{12} + u_{22}) + \beta\delta$. This implies $\beta\delta u_{22} < 1 - (1 - \beta\delta)u_{12}$. But this is contradicted by the 22 condition. This completes the argument that the 22 equilibrium is the unique equilibrium if it exists.

- $\underline{12 \text{ does not coexist with } \pi2.}$ The requirement that $\pi < 1$ for the π2 equilibrium is $(1 - u_{12})(1 - \delta - \beta\delta) < \beta\delta(u_{22} - u_{12})$, which can be rewritten as $1 - u_{12} < \frac{\beta\delta}{1 - \delta}(u_{22} - 1)$, which contradicts the first of the two conditions for the 12 equilibrium. If $\pi = 1$ the two equilibria are equivalent.

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- <u>If 11 and 12 are both equilibria, then so is 1π.</u> It is sufficient to show that

$$u_{21} - u_{22} \leq \frac{\beta\delta}{1 - \delta(1 - \beta)}(u_{21} - 1),$$

  which is the second of the conditions for the 1π equilibrium, as the first condition is implied directly by the existence of the 11 equilibrium. This condition can be rewritten as $u_{21} - u_{22} \leq \frac{\beta\delta}{1-\delta}(u_{22} - 1)$, which is identical to the second of the conditions needed for existence of the 12 equilibrium.

- <u>If 1π exists, so does 11.</u> The 1π case requires two conditions to hold, one of which is $u_{21} - u_{22} \geq \beta\delta(u_{21} - 1)$. But this condition is the only one required for the 11 equilibrium to exist.

- <u>If 11 and 1π are both equilibria, then so is either 12 or $\pi2$.</u> The 12 equilibrium exists if $1 - u_{12} \geq \frac{\beta\delta}{1-\delta}(u_{22} - 1)$, since the second condition under which 12 exists was just shown to be identical to the second condition under which 1π exists. If not, that is, if $1 - u_{12} < \frac{\beta\delta}{1-\delta}(u_{22} - 1)$, we need to show that $\pi2$ exists. This condition can be rewritten as $1 - u_{12} < \frac{\beta\delta}{1-\delta(1-\beta)}(u_{22} - u_{12})$, which implies the second condition for $\pi2$. It remains to show that the first condition for $\pi2$, namely, $1 - u_{12} \geq \beta\delta(u_{22} - u_{12})$, is met. Suppose it is not. Then the only condition for the 22 equilibrium to exist is satisfied. But we showed above that the 22 equilibrium cannot coexist with any other equilibrium; in particular, it cannot coexist with 11 or 1π. This is a contradiction, so the $\pi2$ equilibrium has to exist.

- <u>If 11 and $\pi2$ are both equilibria, then so is 1π.</u> We need to show that the second condition for the 1π equilibrium, $u_{21} - u_{22} \leq \frac{\beta\delta}{1-\delta(1-\beta)}(u_{21} - 1)$, is met (the first one is implied directly since the 11 equilibrium exists). From above, we know that this expression can be rewritten as $u_{21} - u_{22} \leq \frac{\beta\delta}{1-\delta}(u_{22} - 1)$. Now concavity implies that $u_{21} - u_{22} \leq 1 - u_{12}$. We also know, by the second condition for $\pi2$ to exist, that $1 - u_{12} \leq \frac{\beta\delta}{1-\delta(1-\beta)}(u_{22} - u_{12})$, which can be rewritten as $1 - u_{12} \leq \frac{\beta\delta}{1-\delta}(u_{22} - 1)$. Combining these two inequalities yields the desired result.

Going through all possible equilibrium sets, these six facts rule out everything except the six possibilities we claim exist. It is straightforward to verify that these six remaining cases are possible.
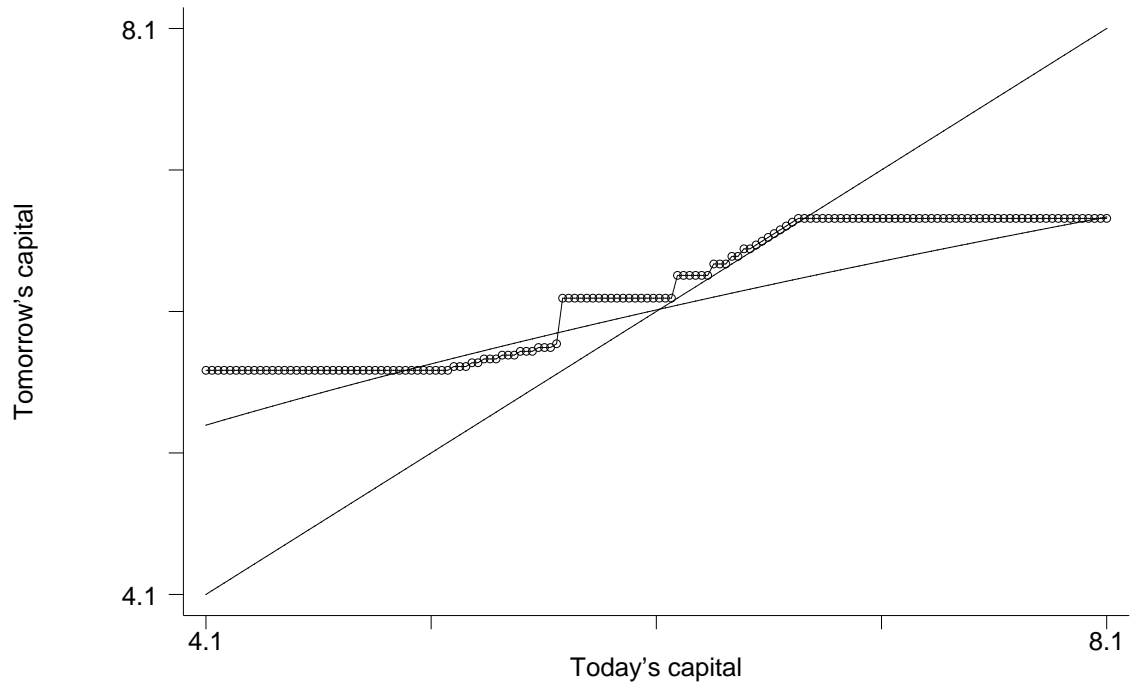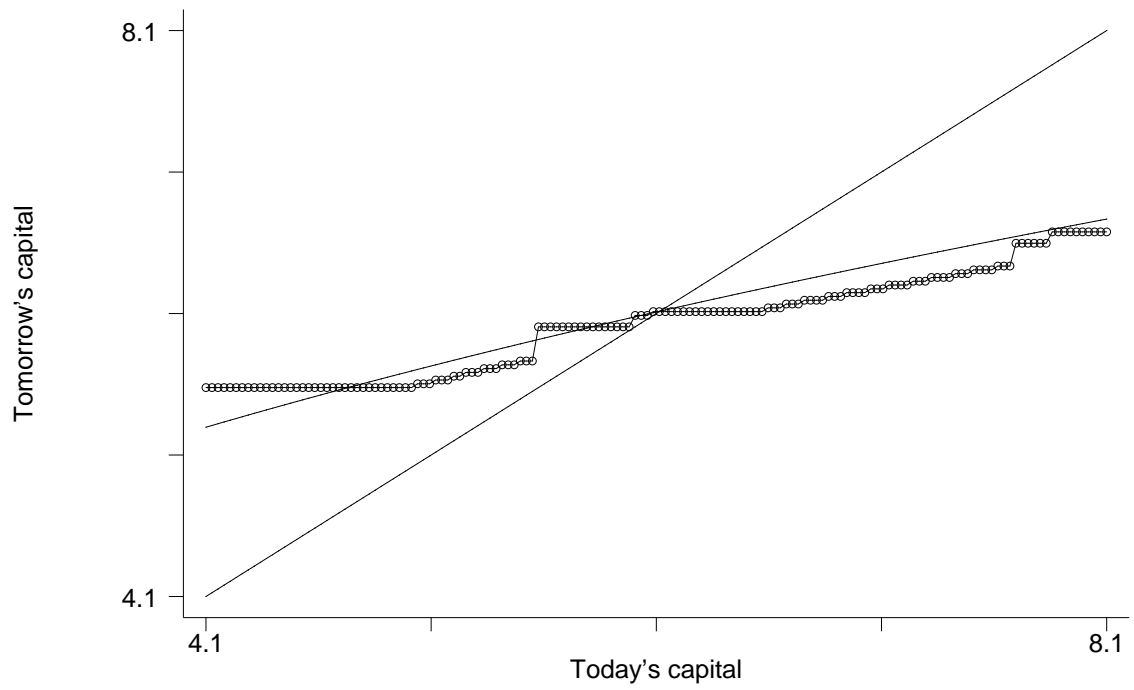
Figure 1

Figure 2

Figure 3