Labor supply in the past, present, and future: a balanced-growth perspective*

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Abstract

What explains how much people work? Going back in time, a main fact to address is the steady reduction in hours worked. The long-run data, for the U.S. as well as for other countries, show a striking pattern whereby hours worked fall steadily by a little below a half of a percent per year, accumulating to about a halving of labor supply over 150 years. In this paper, we argue that a stable utility function defined over consumption and leisure can account for this fact, jointly with the movements in the other macroeconomic aggregates, thus allowing us to view falling hours as part of a macroeconomy displaying balanced growth. The key feature of the utility function is an income effect (of higher wages) that slightly outweighs the substitution effect on hours. We also show that our proposed preference class is the only one consistent with the stated facts. The class can be viewed as an enlargement of the well-known "balanced-growth preferences" that dominate the macroeconomic literature and that demand constant (as opposed to falling) hours in the long run. The postwar U.S. experience, over which hours have shown no net decrease and which is the main argument for the use of "balanced-growth preferences", is thus a striking exception more than a representative feature of modern economies.

Keywords: labor supply, balanced growth, hours worked, Kaldor facts, preferences.
JEL classification: E21, J22, O11, O40.

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1 Introduction

The purpose of this paper is to propose a choice- and technology-based theory for the long-run behavior of the main macroeconomic aggregates. Such a theory—standard balanced-growth theory, specifying preferences and production possibilities along with a market mechanism to be consistent with the data—already exists, but what we argue here is that it needs to be changed. A change is required because of data on hours worked that we document at some length: over a longer perspective—going back a hundred years and more—and looking across many countries, hours worked are falling at a remarkably steady rate: at a little less than half a percentage point per year. Figure 1 illustrates this for a collection of countries.[1] This finding turns

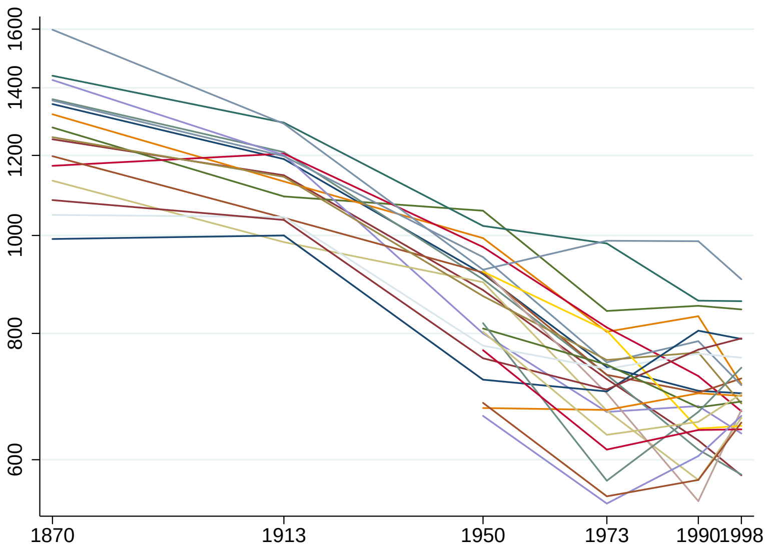


Figure 1: **Average yearly hours worked per capita 1870–1998**

Source: Maddison (2001). The sample includes the following 25 countries: Austria, Belgium, Denmark, Finland, France, Germany, Italy, Netherlands, Norway, Sweden, Switzerland, United Kingdom, Ireland, Spain, Australia, Canada, United States, Argentina, Brazil, Chile, Colombia, Mexico, Peru, Venezuela, Japan. Regressing the log of hours worked on a country fixed effect and year gives a slope coefficient of -0.00462 in the full sample (and -0.00398 for the period 1950–1998). Huberman and Minns (2007) provide similar data.

out to contrast the data in the postwar U.S., where hours are overall well described

[1] We document these facts in greater detail later in the paper using a number of different data sets. The particular data set underlying Figure 1 uses extrapolation for the first data points, except for the U.K., for which there are direct measures.

as stationary, but going back further in time and looking across countries leads one to view the recent U.S. data rather as an exception.

Since the persistent fall in hours worked is not consistent with the preferences and technology used in the standard macroeconomic framework, we alter this framework. Our alteration is very simple and, on a general level, obvious: to rationalize decreasing hours worked we point to steadily increased productivity over very long periods and preferences over consumption and leisure with the feature that income effects on hours exceed substitution effects. As in the case of the standard setting, we however also impose additional structure by summarizing the long-run data as (roughly, at least) having been characterized by *balanced growth*. So on a balanced growth path, our main economic aggregates—hours worked, output, consumption, investment, and the stock of capital—all grow at constant rates. Characterizing the data as fluctuations around such a path may be viewed as a poor approximation, but here we nevertheless do maintain the position that such a characterization is roughly accurate, at least for the last 150 years of data for many developed countries. Hence, we ask: is there a stable utility function such that consumers choose a balanced growth path, with constant growth for consumption, and constant (negative) growth for hours, given that labor productivity grows at a constant rate? We restrict ourselves to time additivity and constant discounting, in line with the assumptions used to derive the standard preference framework. We also restrict attention to the intensive margin of labor supply in our theoretical analysis. We find that there indeed are preferences that do deliver the desired properties and our main result is a complete characterization of the class of such preferences.

The modern macroeconomic literature is based on versions of a framework featuring balanced growth with constant hours worked, to a large extent motivated with reference to postwar U.S. data; see, e.g., Cooley and Prescott (1995). Our main point here is not to take fundamental issue with this practice; in fact, our proposed utility specification in some ways is quantitatively very similar to the preferences normally used. However, for some issues the distinction may be important. As for discussion of hours historically, there is significant recognition in the macroeconomic literature that from a longer-run perspective, hours worked have indeed fallen. For example, several broadly used textbooks actually do point to significant decreases over the longer horizon, often with concrete examples of how hard our grandparents worked; see, e.g., Barro's (1984) book and Mankiw's (2010) latest intermediate text. In a discussion of some significant length, Mankiw actually reminds us of a very well-known short text wherein John Maynard Keynes speculated that hours worked would fall dramatically in the future—from the perspective he had back then (see Keynes, 1930). Keynes thus imagined a 15-hour work week for his grandchildren, in particular, supported by steadily rising productivity. As it turned out, Keynes was wildly off quantitatively, but we would argue that he was right qualitatively (on this issue. . .). Finally, in his forthcoming chapter on growth facts, Jones (2015) also points to the tension between the typical description of hours as stationary and the actual historical data.[2]

From Keynes's U.K. perspective, over the postwar period, and in contrast to the U.S. experience, hours worked actually fell steadily until as recently as circa 1980, at which point they appear to have stabilized; we review the data in some detail in Section 2 below. But perhaps more importantly, the picture that arises from looking at a broader set of countries strengthens the case for falling, rather than constant, hours, and going further back in time reinforces this conclusion. With our eye-balling, at least, a reasonable approximation is actually even more stringent: hours worked are falling at a rate that appears roughly constant over longer periods (though, of course, with swings over business cycles, etc.). This rate is slow—somewhere between 0.3% and 0.5% per year—so shorter-run data will not

[2] Jones writes "A standard stylized fact in macroeconomics is that the fraction of the time spent working shows no trend despite the large upward trend in wages. The next two figures show that this stylized fact is not really true over the longer term, although the evidence is somewhat nuanced".

suffice for detecting this trend, to the extent we are right; to halve the number of hours worked at this rate requires around 175 years.

Turning back to the case of the U.S., over the last more than 150 years, thus, as hours have fallen, output has grown at a remarkably steady rate, mainly interrupted only by the Great Depression and World War II. Moreover, over this rather long period, all the other macroeconomic balanced-growth facts also hold up very well; we review these data briefly in Section 2.3. Thus, as output is growing at a steady rate, hours are falling slowly at a steady rate. The interpretation of these facts that we adopt here is that preferences for consumption and hours belong to the class we define. This preference class is, in fact, very similar to that used ubiquitously in the macroeconomic literature: that defined in King, Plosser, and Rebelo (1988). King, Plosser, and Rebelo showed that the preferences they put forth, often referred to as KPR or, perhaps more descriptively, *balanced-growth* preferences, were the only ones consistent with an exact balanced growth path for all the macroeconomic variables with the restriction to constant hours worked. The class of preferences that we consider in the present paper is thus strictly larger in that it also allows hours worked to change over time at a constant rate along a balanced path.

In compact terms, one can describe the period utility function under KPR as a power function of $cv(h)$, where c is consumption and h hours worked and v is an arbitrary (decreasing) function. What we show in our main Theorem 1 is that the broader class has the same form: period utility is a power function of $cv(hc^{\frac{\nu}{1-\nu}})$, where $\nu < 1$ is the preference parameter that guides how fast hours shrink relative to productivity. In terms of gross rates, if productivity grows at rate γ, then hours grow at rate $\gamma^{-\nu}$, whereas consumption grows at $\gamma^{1-\nu}$. For $\nu > 0$, the factor $c^{\frac{\nu}{1-\nu}}$ captures the stronger income effect: as consumption grows, there is an added “penalty” to working (since v is decreasing). Our preference class obviously nests KPR: KPR corresponds to $\nu = 0$.

Interestingly, our class encompasses some utility functions that are often used in the literature (both in macroeconomics and in other fields). One is another famous functional form of the same vintage as KPR: Greenwood, Hercowitz, and Huffman's (1988) proposed utility function, often referred to as GHH preferences. The GHH class assumes a quasi-linear utility function where utility can be written as a function of c minus an increasing (and convex) function of h. This formulation implies that there is no income effect at all on hours worked. With a judicious choice of v and a $\nu < 0$ we obtain a frequently used case within the GHH class in which the convex function of hours is restricted to be a power function (and the Frisch elasticity is constant). Clearly, without an income effect, hours worked grow under this formulation (so long as productivity grows). GHH preferences are often used in applied contexts (see, e.g., Chetty et al., 2011) because they allow simple comparative statics.

Another well-known case is the utility function proposed in MaCurdy (1981) displaying a constant Frisch elasticity of labor supply and a constant intertemporal elasticity of substitution, where the period utility function is additive in a power function of c and a power function of h. However, unless the function of consumption is logarithmic (a special case of the power function), these preferences are well-known not to be consistent with constant hours worked. We show, again by a judicious choice of v, that our preference class actually includes this case. That is, this class of utility functions is consistent with balanced growth—if one admits that hours can change over time along a balanced path. For shrinking hours, one needs the curvature to be high enough (higher than log curvature), since otherwise the marginal utility value of working an hour will grow: if productivity doubles, the marginal utility of consumption must more than halve, because otherwise it will not be optimal to lower hours.

Another example of how our broadening of the class of balanced-growth preferences may be useful in applications involves curvature. In particular, under our preferences the "consumption curvature", or formally $-u_{cc}(c,h)c/u_c(c,h)$, is an increasing (or decreasing) function of the $hc^{\frac{\nu}{1-\nu}}$ composite. Under KPR preferences, this curvature equal a constant: a preference parameter (usually labeled σ). So, in particular, it is possible that the curvature under our preference specification moves countercyclically, thus displaying higher consumption risk aversion in recessions than in booms. In the cross-section, by the same token, richer households would then be less averse to risk (in the relative sense) and choose riskier portfolios. We briefly discuss this and other possible applications (to growth and business cycles) in Section 8 of our paper.

The paper begins with a data section, Section 2, which looks at hours worked over different time horizons and in different countries. We look at both the intensive and the extensive margin and argue that although there have been noticeable movements in the latter recently, over a longer horizon, essentially the entire fall in total hours is accounted for by the former. Section 2 also briefly reviews the long-run facts for aggregates, with a focus on the United States, in order to then motivate our balanced-growth perspective. In Section 3, and to provide some background, we first briefly discuss the preference class used in the macroeconomic literature to match constant long-run hours and contrast it with a simple example that is actually consistent with falling hours; a well-informed reader might want to skip this section. The theory section of the paper is contained in Section 4 where we lay out the precise balanced-growth restrictions. The main theorem, Theorem 1, is then stated: it states what kind of utility function is needed in order for households to *choose* balanced-growth consumption and labor sequences. The proof of the theorem is in the Appendix. However, the proof relies heavily on two lemmata—one characterizing the implications of balanced-growth choices for the consumption-hours indifference curves and one for consumption curvature—and we discuss those results in some detail in the main text. The theory section also has a Theorem 2, which is straightforward, showing sufficiency of the stated preference class for balanced-growth choices. Section 5 discusses a number of specific functional forms that are useful in applications and comments on some of their properties. Section 6 comments on consumer heterogeneity, a relevant issue since our theory relies on representative-consumer analysis. This section also briefly discusses the cross-sectional wage-hours-wealth data. Section 7 briefly discusses the U.S. postwar data from the perspective of our theory: this data, with its stationary hours, is an exception historically and from an international point of view, so what could explain it? Section 8 concludes.

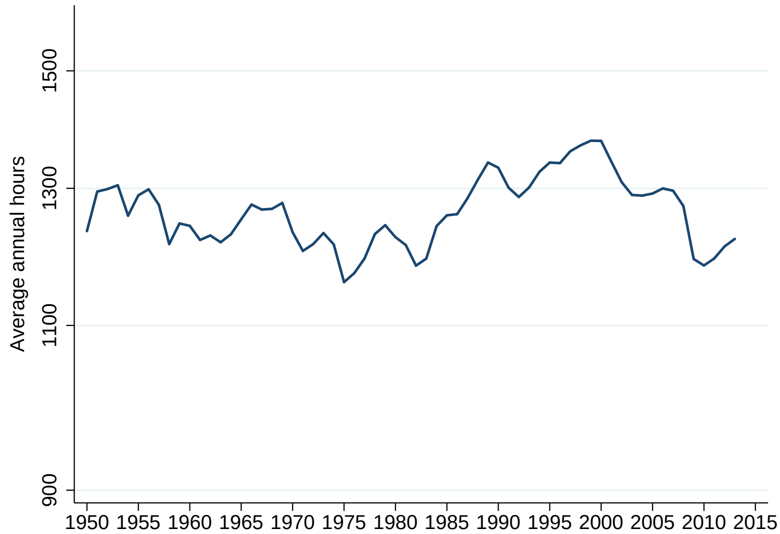


Figure 2: **U.S. average annual hours per capita aged 15–64, 1950–2013**

Notes: Source: GGDC Total Economy Database for total hours worked and OECD for the data on population aged 15–64. The figure is comparable to the ones in Rogerson (2006). Regressing the logarithm of hours worked on time gives an insignificant slope coefficient.

2 Hours worked over time and across countries

We now go over the hours data from various perspectives: across time and space.

2.1 Hours over time

Figure 2 is the main justification for the assumption of constant hours worked maintained in the macroeconomic literature. At least in postwar U.S. data this seems to be a good approximation.

What if we look at some other developed countries? Figure 3 shows hours worked for other selected countries on a logarithmic scale. A horizontal line is no longer a good approximation of the data. A country-fixed effect regression suggests that hours fall at roughly 0.45% per year. To be sure, however, there is significant heterogeneity; Canada, for example, has stationary hours quite like those in the United States.

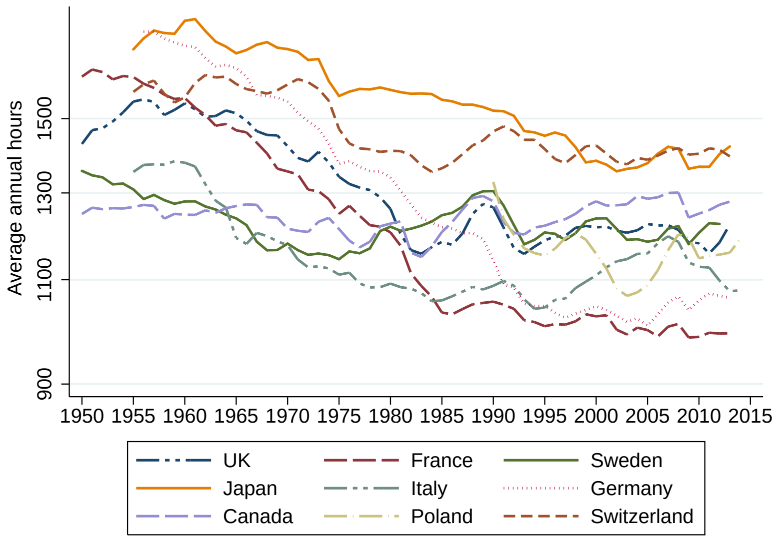


Figure 3: **Selected countries average annual hours per capita aged 15–64, 1950–2015**

Notes: Source: GGDC Total Economy Database for total hours worked and OECD for the data on population aged 15–64. The figure is comparable to the ones in Rogerson (2006). Regressing the logarithm of hours worked on time gives a slope coefficient of -0.00455.

The overall falling hours in Figure 3 are not due to the selection of countries. A complementary Figure C.1 in the Appendix C.1 shows the graph for all countries with available data. Average hours are declining clearly in this unrestricted sample, at roughly 0.34% per year. Hence in the cross-country data of the postwar period the United States and Canada overall rather look like outliers. Interestingly, as Figure C.2 in Appendix C.1 shows, a time-use survey shows decreasing hours worked even for the postwar United States.

From a longer-run perspective, hours worked in the U.S. have also clearly been falling (see Figure 4). We also see that, abstracting from the very large deviations from trend during Great Depression and World War II, hours have been falling at a rather steady rate. Only the period 1980–2000 looks exceptional.

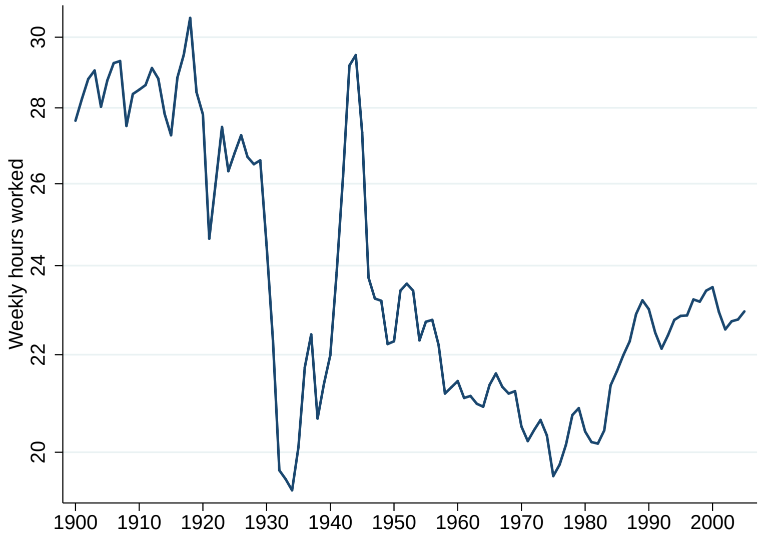


Figure 4: **Weekly U.S. hours worked per population aged 14+, 1900–2005**

Notes: Source: Ramey and Francis (2009). Regressing the logarithm of hours worked on time gives slope coefficient of -0.00285.

Can the falling trend in hours worked be explained by demographics or the rise in schooling? In Figure C.4 in Appendix C.1 we hold hours worked of different age groups constant at their 2005 values and then check whether the observed changes in the age structure can account for the falling hours. The effect implied by the demographic change is non-monotonic and overall very small.[3] Furthermore, Ramey and Francis (2009) also provide data on schooling (time attending school and studying at home). As Figure C.5 in Appendix C.1 shows, average weekly hours of schooling increased by less than two hours in total over the period 1900–2005 and cannot, therefore, account for the drop in hours worked (hence: leisure has increased).

[3] The baby boomers entering prime working age can partially explain the observed increase in hours since the 1980s.

The time trend in total hours worked can be split up into trends in participation rates and trends in hours per worker; in our theory, we focus on the latter. Figure 5 shows that hours per employed in the U.S. declined at a remarkably constant rate, including during the postwar period. This is indeed a remarkably robust fact over time and across countries though the rate of decline differs across countries (see Figure C.3 in Appendix C.1). Figure C.6 and C.7 in Appendix C.1 show this split in hours per worker and the participation rate again for the U.S. in the postwar period as well as over the last century. In other words: hours in the postwar U.S. are only relatively stable because the participation rate increased steeply.

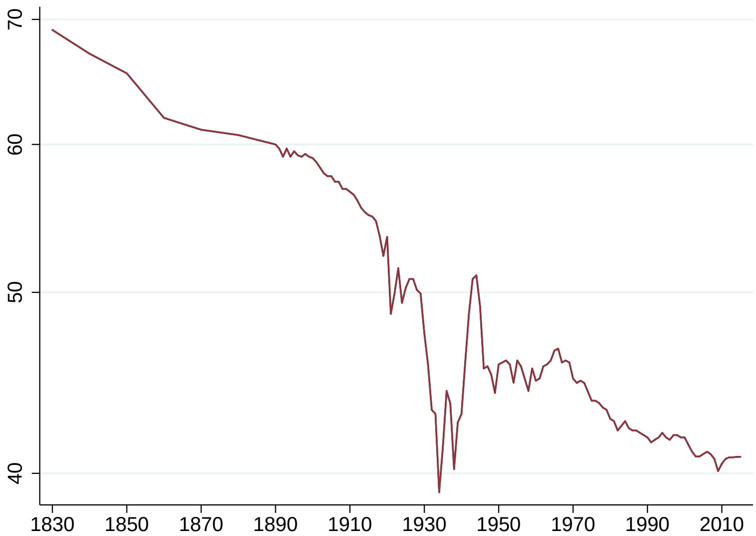


Figure 5: **U.S. weekly hours worked per worker in nonfarm establishments 1830–2015**

Source: Average weekly hours data for 1830–80: Whaples (1990, Table 2.1). 1890-1970: Historical Statistics of the United States: Colonial Times to 1970 (Series D765 and D803). 1970–2015: Statistical Abstract of the United States the number for nonfarm establishments. This graph shows an updates series of the data in Greenwood and Vandenbroucke (2008). Regressing the log of hours on a constant and year gives a slope coefficient of -0.00315 in the full sample (and -0.00208 for the years 1970–2015).

To sum up: over 100+ years, hours have been falling in all developed countries. In the postwar data hours are still falling in most countries. In countries where they are rather stable, like Canada or the U.S., they are stable only because the participation rate increased quite dramatically. Hours per worker show a clear downward trend in all countries. Participation rates do not show a clear trend over time in developed countries. Hence we conclude that, purely in terms of trend extrapolation, if the participation rate does not increase further in the U.S., hours will continue to fall. In fact, since the Great Recession, the participation rate fell, as did hours worked per working-age population.

2.2 Hours worked in the cross-section of countries

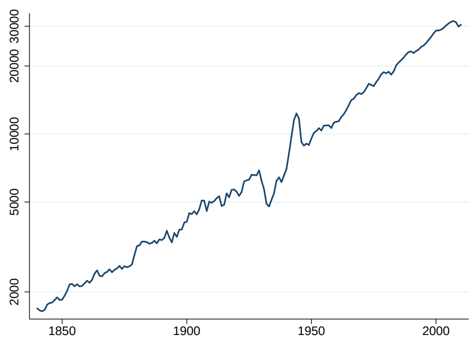
In the cross-section of countries, our theory predicts that labor productivity (or GDP per capita) should be negatively correlated with hours worked. Winston (1966) establishes such a negative relationship in a sample of 18 countries and estimates the elasticity of hours worked with respect to GDP per capita to be -0.107.[4] Bick, Fuchs-Schuendeln, and Lagakos (2015) document this negative correlation for a larger sample that includes developing countries. Figure C.8 and Figure C.9 in Appendix C.1 shows this negative correlation in the postwar data for the pooled sample and the years 1955 and 2010 separately.

Finally, in Figure C.10 Appendix C.1 we focus on the 21 countries with data for 1955–2010 and look at the correlation in the growth rates in labor productivity and hours worked over these 55 years. The figure shows again that hours fell for most of the countries. Moreover, with the exception of South Korea, labor productivity growth is clearly negatively related with the growth rate in hours worked.

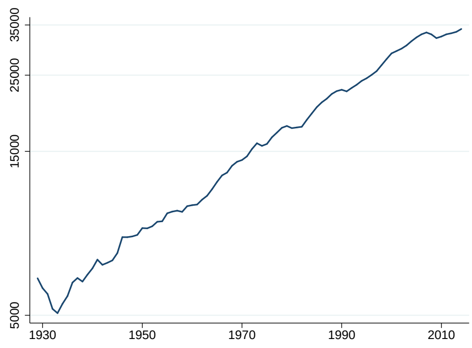
2.3 The balanced-growth facts

Lastly, and for completeness, we now review the basic "stylized facts of growth" for the United States. These data have been instrumental in guiding the technology and preference specifications in macroeconomic theory.

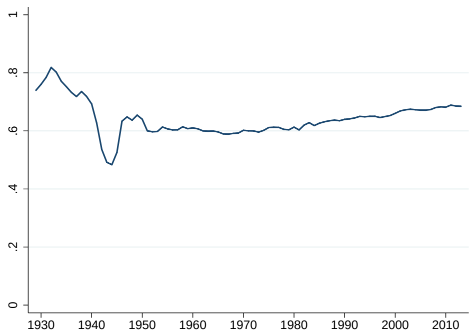
[4] With a labor productivity growth of 2.5 percent per year this slope coefficient suggests that hours worked decrease at 0.26 percent per year.



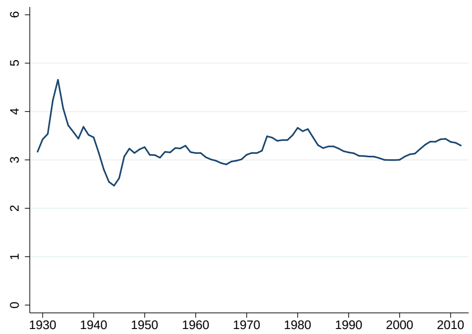
(a) GDP per capita



(b) Consumption per capita



(c) Consumption-output ratio



(d) Capital-output ratio

Figure 6: Balanced growth

Source: BEA and Maddison project.

Figure 6a and 6b show how output and consumption grew over the decades at a very steady rate. Figure 6c and 6d show that the consumption-output ratio and the capital-output ratio remained remarkably stable. (Figure C.11 in Appendix C.1 shows the capital-output ratio over an even longer time horizon and an additional balanced-growth fact often imposed in the macroeconomic literature: constant factor income shares.) Our main take-away message from Figure 6 is that—in the style of Kaldor (1961)—we would like to impose restrictions on our macroeconomic framework such that it is consistent with these facts.

3 Standard utility and a contrasting example

We now proceed to discuss whether the data we illustrated above can be rationalized based on a stable utility function over consumption and leisure. To do this, we begin in Section 3.1 by reminding the reader of the utility functions ordinarily used in macroeconomic analyses when restricting attention to balanced growth paths with constant hours. We thus present King, Plosser, and Rebelo's (1988) formulation and a parametric example which encompasses all the most commonly used functions. Furthermore, we give an example in Section 3.2 that is outside this class and that admits hours falling at a constant rate along a balanced growth path. We then proceed with the general analysis in the following sections.

3.1 King, Plosser, and Rebelo (1988)

Consider time-additive preferences with a period utility function $u(c,h)$, where c is consumption and h hours worked. King, Plosser, and Rebelo (1988; KPR for short) show that balanced growth with constant hours worked obtain if and only if u can be written as

$$u(c,h) = \begin{cases} \frac{(c \cdot v(h))^{1-\sigma}-1}{1-\sigma} & \text{if } \sigma \neq 1 \\ \log(c) + \log v(h), & \text{if } \sigma = 1. \end{cases} \tag{1}$$

The KPR class, sometimes referred to as "balanced-growth preferences", has dominated the applied macroeconomic literature; in this literature, it is considered paramount to use a framework that is consistent with a balanced growth path.

Within the KPR class, two special cases stand out. One is the Cobb-Douglas case: $u(c,h) = (c(1-h)^\kappa)^{1-\sigma}/(1-\sigma)$ for $\sigma \neq 1$ and (replacing the $\sigma = 1$ case) $u(c,h) = \log(c) + \kappa \log(1-h)$. The Cobb-Douglas case, which is obtained by setting $v(h) = (1-h)^\kappa$ in (1), restricts the elasticity of substitution between consumption and leisure to be one. This case, furthermore, is part of the Gorman class, i.e., the marginal propensities to consume and work are independent of wealth.

The second frequently used case of KPR preferences is

$$u(c,h)=\log{(c)}-\psi\frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}, \tag{2}$$

which is obtained by setting $\sigma = 1$ and $v(h) = \exp\left(-\psi\frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}\right)$. The parameter $\theta > 0$ is then the (constant) Frisch elasticity, i.e., the percentage change in hours when the wage is changed by 1 percent, keeping the marginal utility of consumption (or wealth) constant.

One can actually nest the two above cases as

$$u(c,h)=\frac{c^{1-\sigma}\left(1-ah^{b}\right)^{d}-1}{1-\sigma}, \tag{3}$$

which, for future reference, typically will be parameterized using ψ, θ, and κ, as

$$a=\frac{\psi(1-\sigma)}{1+\frac{1}{\theta}},\quad b=1+\frac{1}{\theta},\quad \text{and}\quad d=(1-\sigma)\kappa.$$

The functional form in (3) is obtained by setting $v(h) = \left(1 - ah^b\right)^{\frac{d}{1-\sigma}}$ in (1).[5] It is straightforward to verify that the Cobb-Douglas case obtains when $\psi = 1/(1-\sigma)$ and $\theta \to \infty$. One can also show that the Frisch elasticity is constant when $\kappa = \sigma/(1-\sigma)$, which delivers

$$u(c,h)=\frac{c^{1-\sigma}\left(1-\psi(1-\sigma)\frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}\right)^{\sigma}-1}{1-\sigma}. \tag{4}$$

With $\sigma \to 1$, the the formulation in (4) reduces to (2) as a special case. These two cases, (2) and (4), are considered in Trabandt and Uhlig (2011), who also show that these are the only functional forms within KPR admitting a constant Frisch elasticity.

[5] We again assume that for $\sigma = 1$, the utility function is given by the limit, or formally $\lim_{\sigma\to 1} u(c,h)$.

We will return to the "general special case" in (3) below in the context of falling hours.

3.2 An example with falling hours

The KPR class does not admit hours falling at a constant rate. So suppose we look at a case outside their class: the rather familiar function

$$u(c,h) = \frac{c^{1-\sigma}-1}{1-\sigma} - \psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}},$$

which was proposed in MaCurdy (1981). Notice that this case is a generalization—allowing different consumption curvature than the log case—of the most commonly used constant-Frisch formulation in KPR: (2). A consumer facing a wage rate w_t at time t would thus have an intratemporal first-order condition reading

$$w_t c_t^{-\sigma} = \psi h_t^{\frac{1}{\theta}}.$$

Is this equation consistent with balanced growth, in particular with hours falling at a constant rate? Suppose that wages grow at rate $\gamma > 1$ and that consumption grows at rate γ_c, with hours growing at γ_h, all in gross terms. For the first-order condition to hold at all points in time we then need that

$$\gamma \gamma_c^{-\sigma} = \gamma_h^{\frac{1}{\theta}}.$$

On the type of balanced growth path considered in typical macroeconomic models, we would have $\gamma = \gamma_c$, which is indeed consistent with this equation and then implies that hours must grow at gross rate $\gamma_h = \gamma^{(1-\sigma)\theta}$. However, unless $\sigma = 1$ (logarithmic curvature in consumption) this suggestion would not be consistent with the budget or the aggregate (closed-economy) resource constraint as we would have $w_t h_t$ growing at a different rate than c_t. Rather, for labor income and consumption to grow at the same rate we need $\gamma\gamma_h = \gamma_c$. Inserting this into the previous equation instead we obtain $\gamma^{1-\sigma} = \gamma_h^{\frac{1}{\theta}+\sigma}$, so that $\gamma_h = \gamma^{\frac{\theta(1-\sigma)}{1+\theta\sigma}}$. We also see from this example that hours will be falling over time if $\sigma > 1$. Consumption will thus grow at rate $\gamma_c = \gamma^{\frac{1+\theta}{1+\theta\sigma}}$.

Turning to intertemporal considerations, the Euler equation here is a standard one, since $u(c, h)$ is additive; clearly, under a constant interest rate, it can be met for a consumption path growing at a constant rate. Thus, we conclude that, at least based on this utility function, it seems possible to rationalize falling hours worked along a balanced growth path.

4 Theory

4.1 Balanced growth: technology and preferences

We now begin to set up our formal analysis. We will first state the balanced-growth restrictions from the perspective of the aggregate resource constraint in a closed economy. The workhorse macroeconomic framework has a final-good resource constraint given by

$$K_{t+1} = F\left(K_t, A_t h_t L_t\right) + (1-\delta)K_t - L_t c_t, \tag{5}$$

where capital letters refer to aggregates and lower-case letters per-capita values, and $F\left(K_t, A_t h_t L_t\right)$ is a neoclassical production function. Here, L is population, h is hours worked per capita and δ the depreciation rate. Growth is of the labor-augmenting kind, because of the Uzawa theorem.[6] We thus assume constant exogenous technol-

[6] Recently, Grossman, Helpman, Oberfield, and Sampson (2015) discusses an interesting exception to the Uzawa theorem.

ogy and population growth, i.e.,

$$A_t = A_0\gamma^t, \text{ and } L_t = L_0\eta^t. \tag{6}$$

Turning to preferences, we assume that they are additively separable over time with a constant discount factor β. Quite importantly, and in line with the KPR setting, the instantaneous utility, $u(c_t, h_t)$, is assumed to be stationary. Then households (whether infinitely or finitely lived) maximize

$$\cdots + u(c_t, h_t) + \beta u(c_{t+1}, h_{t+1}) + \ldots \tag{7}$$

subject to a budget constraint

$$a_{t+1} = (1 + r_t)a_t + h_t w_t - c_t \tag{8}$$

and, usually, a time constraint

$$h_t + l_t = 1, \tag{9}$$

(where l denotes leisure per capita and both h and l are non-negative). In the following, however, we will focus on interior solutions so this time constraint will not be used actively.

A balanced-growth path for this economy is a time path along which K and c grow at constant rates. Feasibility of such a path thus requires $\frac{A_{t+1}}{A_t}\frac{h_{t+1}}{h_t}\frac{L_{t+1}}{L_t} = \frac{L_{t+1}}{L_t}\frac{c_{t+1}}{c_t} = \frac{K_{t+1}}{K_t}$ (see (5)). Hence, since $\frac{A_{t+1}}{A_t} = \gamma$ and $\frac{L_{t+1}}{L_t} = \eta$ a balanced-growth path implies a constant $\frac{h_{t+1}}{h_t}$.

On a balanced growth path where labor productivity (alternatively, the real wage per hour) changes at constant gross rate $\gamma > 0$, we need to have consumption grow at the same rate as labor income. The derivations above led to $g_c = \gamma g_h$, where g_c is the gross growth rate of consumption and g_h that of hours worked. We thus seek preferences such that g_c and g_h are determined uniquely as a function of the growth rate in (real) wages. Thus, we parameterize preferences with a constant ν so that $g_c = \gamma^{1-\nu}$ and $g_h = \gamma^{-\nu}$.[7] A value of ν greater (smaller) than zero would then correspond to the income effect of increasing productivity on hours being stronger (weaker) than the substitution effect. The special case $\nu = 0$ is of interest but we will mainly focus on $\nu \neq 0$; $\nu = 0$ is the standard case, where hours will be constant on a balanced growth path.

Thus, a balanced growth path is one where, for all t, $c_t = c_0\gamma^{(1-\nu)t}$ and $h_t = h_0\gamma^{-\nu t}$, for some values c_0 and h_0. One can think of c_0 as a free variable here, determined by the economy's, or the consumer's, overall wealth, with h_0 pinned down by a labor-leisure choice given c_0.

In the following we are interested in an interior solution of the consumption and labor supply decision (i.e., $c_t > 0$, $1 > h_t > 0$) that is consistent with a balanced growth path: we confine attention to the intensive margin of labor supply.[8] Such an interior solution requires utility to be strictly increasing in consumption and strictly decreasing in hours worked as well as some additional regularity conditions we will comment on further below. Two first-order conditions are relevant for the consumer's optimization. The labor-leisure choice is characterized by

$$-\frac{u_2(c_t, h_t)}{u_1(c_t, h_t)} = w_t,$$

where w_t, the return from working one unit of time, grows in the long-run at rate γ: $w_t = w_0\gamma^t$.[9] On a balanced growth path we thus need this condition to hold for all t. In our theorem below, we will also require that preferences admit a balanced growth

[7] With $\nu \geq 1$ the theory would predict decreasing (or constant) consumption as the wage rate increases; we rule this case out.

[8] We comment on the extensive margin in Section 6 below.

[9] In a decentralized equilibrium, this return denotes the individual wage rate including potential taxes and transfers. Similarly, the return on saving we discuss below should be taken to be net of taxes and transfers.

path for all $w_0 > 0$. That is, we are looking for preferences that imply first-order conditions that admit a balanced path for consumption and hours at growth rates $\gamma^{1-\nu}$ and $\gamma^{-\nu}$, respectively, regardless of the (initial) level of the wage rate relative to consumption.

The intertemporal (Euler) equation reads

$$\frac{u_1(c_t, h_t)}{u_1(c_{t+1}, h_{t+1})} = \beta(1 + r_{t+1}),$$

where r is the return on saving and $\beta > 0$ the discount factor. If the economy grows along a balanced path, then we would like this condition to hold for all t, and we need the right-hand side to be equal to an appropriate constant, a constant that moreover may depend on the rate of growth of consumption and hours. We will denote this constant R and discuss its dependence on c, h, and γ below. In the subsequent analysis, we will switch from sequence to functional notation. Thus we leave out t subscripts and instead specify the balanced-growth conditions as a requirement that the paths of all the variables start growing from arbitrary positive values (save for those nonlinear restrictions relating the variables to each other that are implied by the equilibrium conditions): they can be scaled arbitrarily.

4.2 Balanced growth using functional language

Proceeding toward our formal analysis, now note that our balanced-growth path requirements on the utility function can be expressed as follows.

Assumption 1. *The utility function u has the following properties: for any $w > 0$, $c > 0$, and $\gamma > 0$, there exists an $h > 0$ and an $R > 0$ such that, for any $\lambda > 0$,*

$$-\frac{u_2\left(c\lambda^{1-\nu}, h\lambda^{-\nu}\right)}{u_1\left(c\lambda^{1-\nu}, h\lambda^{-\nu}\right)} = w\lambda, \tag{10}$$

and

$$\frac{u_1\left(c\lambda^{1-\nu}, h\lambda^{-\nu}\right)}{u_1\left(c\lambda^{1-\nu}\gamma^{1-\nu}, h\lambda^{-\nu}\gamma^{-\nu}\right)} = R, \tag{11}$$

where $\nu < 1$.

That is, we must be able to scale variables arbitrarily, but of course consistently with the balanced rates, and still satisfy the two first-order conditions. The scaling is accomplished using λ (for wages/productivity), $\lambda^{1-\nu}$ (for consumption), and $\lambda^{-\nu}$ (for hours) in these conditions. Our main theorem below will thus characterize the class of utility functions u consistent with these conditions. Our theorem will not provide conditions on convexity of the associated maximization problem (of the consumer, or a social planner); obviously, however, conditions must be added such that the first-order conditions indeed characterize the solution. We briefly discuss this issue in Section 4.3.5.

4.3 The main theorem

Our main theorem states what restrictions on the utility function are necessary for generating balanced growth.

Theorem 1. *If* $u(c,h)$ *is twice continuously differentiable and satisfies Assumption 1, then (save for additive and multiplicative constants) it must be of the form*

$$u(c,h) = \frac{\left(c \cdot v\left(hc^{\frac{\nu}{1-\nu}}\right)\right)^{1-\sigma} - 1}{1-\sigma},$$

for $\sigma \neq 1$*, or*

$$u(c,h) = \log(c) + \log\left(v(hc^{\frac{\nu}{1-\nu}})\right),$$

where v *is an arbitrary, twice continuously differentiable function.*

The proof relies crucially on two lemmata, one characterizing the marginal rate of substitution (MRS) function between c and h and one characterizing the curvature with respect to consumption: the relative risk aversion in consumption (RRA_c) function. The proof then uses these lemmata to derive the final characterization. The proofs of the lemmata and of how to use them to complete the proof of the theorem are contained in Appendix A.1. However, we will state and comment on the lemmata, as they are of some independent interest, as well as on the overall method of proof.

4.3.1 The consumption-hours indifference curves

We thus begin with the following lemma:

Lemma 1. *If $u(c, h)$ satisfies (10) for all $\lambda > 0$, and for an arbitrary $c > 0$ and $w > 0$, then its marginal rate of substitution (MRS) function, defined by $u_2(c, h)/u_1(c, h)$, must be of the form*

$$\frac{u_2(c,h)}{u_1(c,h)} = c^{\frac{1}{1-\nu}} q(hc^{\frac{\nu}{1-\nu}}), \tag{12}$$

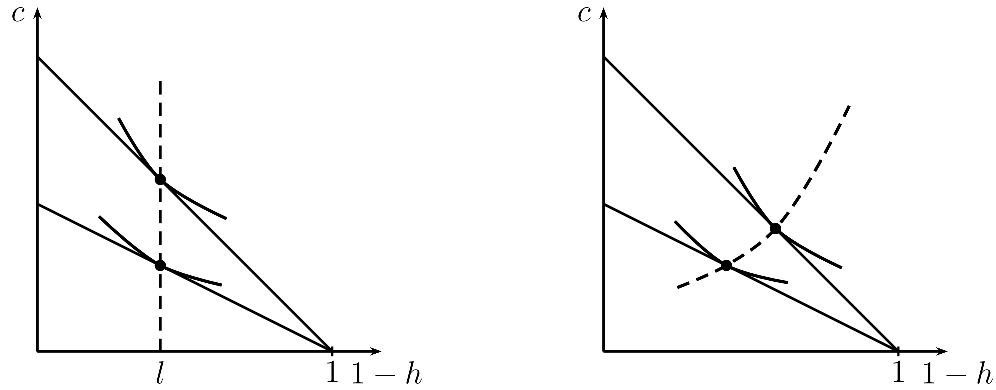
for an arbitrary function q.

This lemma characterizes the shape of the within-period indifference curves. Notice here that, in the long run, $hc^{\frac{\nu}{1-\nu}}$ will be constant so that the argument of q will not change over time. The proof technique for Lemma 1 is very similar to that for Euler's theorem.

The indifference curves are illustrated with the following sequence of graphs. In Figure 7, we see the KPR indifference curves to the left and a case with $\nu > 0$ to the right, with consumption and leisure on the axes.[10] Clearly, with $\nu > 0$, a higher labor productivity implies more leisure: the income effect exceeds the substitution effect.

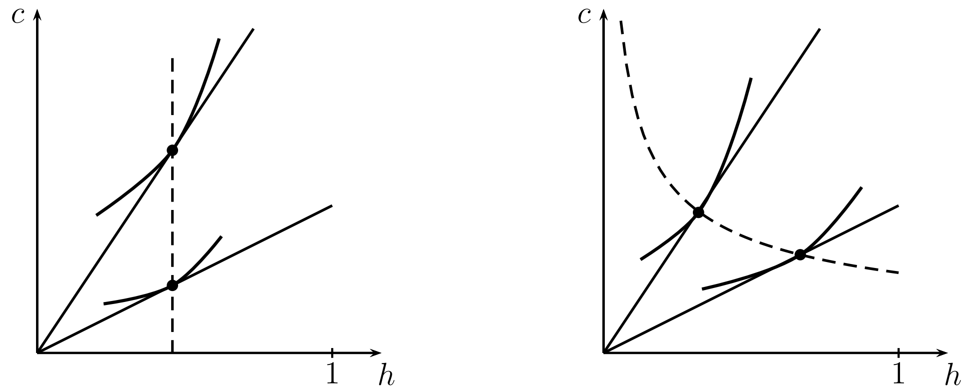
These same preferences can equivalently be depicted with consumption and hours on the axes, as in Figure 8. As in the previous figure, the KPR case is to the left

[10] For simplicity, we abstract from non-labor income in Figure 7.



(a) $\nu = 0$ (b) $\nu > 0$

Figure 7: The consumption-leisure trade-off, I

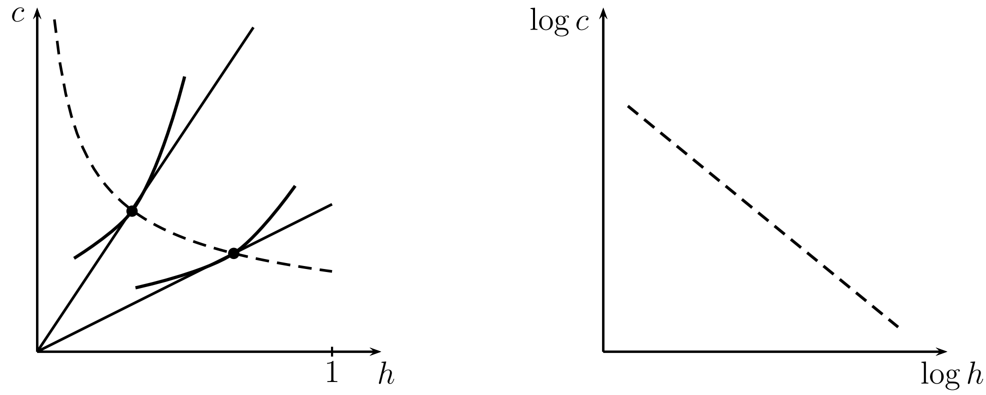


(a) $\nu = 0$ (b) $\nu > 0$

Figure 8: The consumption-leisure trade-off, II

and has constant hours worked, whereas in the right-hand side panel hours decline with higher labor productivity.

Finally, Figure 9 takes the right-hand side graph from the previous figure and puts it on the left. On the right, now, we see that same combination of points but on log scales for both the axis. Here, the expansion path (dashed line) is linear, and that is the defining characteristic of the indifference curves in Lemma 1: that is the precise way in which the income and substitution effect have to be related in the



(a) in levels

(b) log scale

Figure 9: The consumption-leisure trade-off, III

class of utility functions that delivers balanced growth.[11]

4.3.2 Curvature

Next, let us characterize curvature of u with respect to c with Lemma 2.

Lemma 2. *Under Assumption 1, the relative risk aversion in consumption (RRA_c), $\frac{-cu_{11}(c,h)}{u_1(c,h)}$, must satisfy*

$$-\frac{cu_{11}(c,h)}{u_1(c,h)} = p(hc^{\frac{\nu}{1-\nu}})$$

for an arbitrary function p.

As for the previous lemma, let us point out that in the long run, i.e., along a balanced-growth path, $hc^{\frac{\nu}{1-\nu}}$ is constant. Thus, the $\mathrm{RRA_c}$ will be constant. However, in general, its long-run level is endogenous, and over shorter time horizons, it will not be constant.

The proof of the lemma is straightforward: it involves differentiation of the Euler equation with respect to λ, the use of Lemma 1, and some manipulations.

[11] The slope in Figure 9b is $-\frac{1-\nu}{\nu}$. For $\nu < 0$, the substitution effect would be stronger and hours/effort would increase as the wage rises; then the right-hand side panel of Figure 9 would depict a straight line with a positive slope.

The term relative risk aversion here is used for convenience; it is appropriate only to the extent we consider a gamble where hours are not allowed to adjust. As Swanson (2012) has shown, the appropriate risk aversion concept in typical applied dynamic models with valued leisure is based on the value function.

4.3.3 The proof structure and some comments

The structure of the overall proof, based on the lemmata, is as follows. Our description is in two steps that are similar in nature. First, we use Lemma 2 to integrate over c to obtain a candidate for u_1; this can be accomplished straightforwardly since the left-hand side of the lemma can be expressed as the derivative of $\log u_1$ with respect to $\log c$. Now note that integration with respect to one variable delivers an unknown function (a "constant") of the other variable. This function can then be restricted by comparison with the characterization in Lemma 1 (a "cross-check").

Second, after the first integration and cross-checking, with its implied restrictions, we integrate again with respect to c from the obtained u_1 to deliver a candidate for u. Then, as in the previous step, another function of h appears and it too needs to be cross-checked with Lemma 1 and thus further restricted. This, then, completes the proof.

Notice that, although we were motivated by data displaying increasing productivity growth and falling hours, the proof does not assume $\gamma > 1$ or $\nu \geq 0$. Potentially, the model could thus generate an increasing h at a constant rate as productivity increases steadily, and we shall see an example of this below.

Furthermore, to our surprise, we did not see a full proof of the KPR result in the literature.[12] In particular, in the proofs we have looked at, the fact that the RRA_c is constant along a balanced path is taken to mean that this constant is exogenous (i.e., given by a preference parameter σ and independent of h). This is a

[12] We would be very grateful if someone could point us to a proof somewhere, because we may well have missed it.

correct presumption but nontrivial to prove, and it is dealt with in our proof in the Appendix A.1.

4.3.4 Sufficiency

We now provide the converse of the theorem above: with the utility function in the specified class, the first-order conditions for optimization will be consistent with balanced growth.

Theorem 2. *Assume that $\nu < 1$. If $u(c,h)$ is given by*

$$u(c,h) = \frac{\left(c \cdot v\left(hc^{\frac{\nu}{1-\nu}}\right)\right)^{1-\sigma} - 1}{1-\sigma},$$

for $\sigma \neq 1$, or

$$u(c,h) = \log(c) + \log\left(v(hc^{\frac{\nu}{1-\nu}})\right),$$

where v is an arbitrary, twice continuously differentiable function, then it satisfies Assumption 1.

Since this proof is much less cumbersome than that for the main theorem, and since it involves the manipulations necessary in applied work based on the preference class we identify here, we include it in the main text.

Proof. Straightforward differentiation delivers

$$u_1(c,h) = \frac{1}{c}\left(1 + \frac{\nu}{1-\nu}\frac{v'(hc^{\frac{\nu}{1-\nu}})}{v(hc^{\frac{\nu}{1-\nu}})}hc^{\frac{\nu}{1-\nu}}\right)\left(c \cdot v\left(hc^{\frac{\nu}{1-\nu}}\right)\right)^{1-\sigma}$$

and

$$u_2(c,h) = \frac{1}{h}\frac{v'(hc^{\frac{\nu}{1-\nu}})}{v(hc^{\frac{\nu}{1-\nu}})}hc^{\frac{\nu}{1-\nu}}\left(c \cdot v\left(hc^{\frac{\nu}{1-\nu}}\right)\right)^{1-\sigma}.$$

Dividing the latter by the former we obtain

$$\frac{u_2(c,h)}{u_1(c,h)} = \frac{c}{h} \frac{\frac{v'(hc^{\frac{\nu}{1-\nu}})}{v(hc^{\frac{\nu}{1-\nu}})} hc^{\frac{\nu}{1-\nu}}}{1 + \frac{\nu}{1-\nu} \frac{v'(hc^{\frac{\nu}{1-\nu}})}{v(hc^{\frac{\nu}{1-\nu}})} hc^{\frac{\nu}{1-\nu}}}.$$

By multiplying c by $\lambda^{1-\nu}$ and h by $\lambda^{-\nu}$ we obtain that this expression increases by a factor λ. We have thus reproduced the first part of Assumption 1, i.e., the intratemporal first-order condition on a balanced-growth path.

By evaluating $u_1(c,h)/u_1(c\gamma^{1-\nu}, h\gamma^{-\nu})$, we obtain $\gamma^{\sigma(1-\nu)}$, i.e., an expression that is independent of c and h and hence c and h can be scaled arbitrarily. By letting $R = \gamma^{\sigma(1-\nu)}$ we therefore see that also the second condition of Assumption 1 is verified.

■

4.3.5 Utility maximization under explicit constraints

Our two theorems together state necessary and sufficient conditions for our utility function to be consistent with balanced growth as represented by an interior solution given by the first-order conditions in Assumption 1. The theorems are thus designed strictly to characterize preferences. Whether an exact balanced growth path exists, as a competitive equilibrium or the solution to a planning problem, with preferences in the defined class is a different, though of course closely related, question. The answer depends on features of the constraints facing the consumer/planner. For example, if the marginal product of capital is not high enough for the given productivity growth rate when the capital input is at zero, the balanced-growth version of the Euler equation could not be satisfied. Moreover, to ensure sufficiency of a maximum based on the first-order conditions requires not only features of the constraints but also additional assumptions on preferences. We now briefly and informally discuss some of the issues that come up when discussing the more general existence question. As a preview and summary, let us simply say that the main additional restrictions are those normally assumed on constraints (such as non-emptiness, compactness, and convexity) and preferences (such as monotonicity and concavity).

Let us begin by pointing out that the conditions in Assumption 1 formally arise from a consumer maximizing utility subject to a budget constraint where prices (w, including its scaled-up values, and R) are treated as exogenous. As such, it is rather general and could straightforwardly be applied to models where market allocations are not necessarily optimal (such as with taxes or an overlapping-generations structure). If one took a planner's perspective, however, the same first-order conditions would arise with w represented by a marginal product of labor and R by a marginal product of capital net of depreciation. Both these would then be functions of endogenous variables (capital and hours) but, on a balanced path, these would be constant if the production function is homogeneous of degree one, and therefore it is appropriate to regard them as arbitrary constants and hence our theorems can be applied directly for a planning problem as well. However, whether appropriate such constants exist for given technologies, and what they are, precisely requires some additional discussion.

Since the constraints can differ depending on the specific application, we will not present an additional theorem here but rather discuss three key issues—existence, interiority, and sufficiency, each of which leads to potential pitfalls—and mention some possible further restrictions on v. Starting with existence, as already alluded to, one needs assumptions on technology to ensure that the first-order conditions be met on balanced growth paths. Such assumptions are usually ensured with Inada conditions, on production as well as preferences. To ensure existence of a solution to a given maximization problem, one would typically make sure that the utility function is continuous and the constraint set compact for an appropriately chosen topological space. Here, a variety of standard regularity assumptions could be invoked on our v and on the constraints.

Turning to interiority, this is a presumption we have been working with throughout. Clearly, for the first-order conditions to make sense we need utility to be monotonic: (strictly) increasing in c and (strictly) decreasing in h. This is accomplished with the assumptions $v(x) > -\frac{\nu}{1-\nu}v'(x)x$ and $v'(x) < 0$ for all x.[13] Moreover, one needs to ascertain that corner solutions not apply; to rule such cases out, Inada conditions could again be used. E.g., to rule out zero hours for a consumer maximizing utility, $u_2(c, 0) = 0$ for all $c > 0$ would be useful; for a planner, an additional possibility is to assume $F_2(K, 0) = \infty$ for all $K > 0$. However, corner solutions could be relevant and we do not want to argue generally that strong assumptions on v be made to ensure Inada conditions. It seems reasonable that zero hours are in fact chosen for some consumers (with high enough wealth/low enough productivity); moreover, we would like to think of hours in time units and then there is also a natural upper bound on hours, one that will be reached for a low enough productivity/wealth level; such bounds may have been relevant historically, in fact.[14]

Second, one needs conditions for first-order conditions to be sufficient. In a static context of labor-leisure choice, first-order conditions together with quasi-concavity of $u(c, h)$ constitute a typical sufficiency condition. In a dynamic model, matters are somewhat more complicated. Stokey and Lucas (1989, Theorem 4.15) show how to prove sufficiency of the first-order conditions in an infinite horizon framework together with a transversality condition, provided that the "return function", denoted $F(k, k')$ there, is jointly concave in k and k'. Here, the return function involves h as well and $F(k, k', h)$ then denotes $u(c, h)$, where c has been replaced, using a resource or budget constraint, by a concave function of (k, h, k'). The Stokey-Lucas proof extends straightforwardly to the case where h is included, so long as one can show that F is now concave in (k, h, k'). The assumption that $u(c, h)$ is (strictly) concave

[13] However, one would not want to rule out balanced growth paths where these assumptions are met for some, but not all, values of x.

[14] With an upper bound on working time, we also need to assume $(\gamma - 1)\nu \geq 0$.

in (c, h) suffices here to obtain (strict) concavity of F. The transversality condition is derived straightforwardly exactly as in Stokey and Lucas's treatment. In sum, given a $v(x)$ that, jointly with constraint set, is such that $u(c, h)$ becomes strictly concave, standard analysis based on first-order conditions can be used.

Turning to examples, consider a price-taking consumer, i.e., a constraint in the form of an affine constraint in the choice variables each period. Let us first point out that the MaCurdy (1981) function then satisfies all the above conditions including, to the extent $\sigma > 1$, an Inada condition for consumption. In Section 5.3.3 below we look at another concrete special case—that where $\sigma = 1$—for which we are also able to solve for full transitional dynamics in closed form under appropriate assumptions on technology; there as well, monotonicity and concavity are verified, along with the existence of a maximum. More generally, one can imagine many other specific assumptions on v under which the same conditions go through. It is of course also straightforward to produce assumptions on v such that monotonicity and concavity are not met. More nontrivially, it is possible to find preferences for Assumption 1, monotonicity, and concavity are all met but that at the same time do not allow falling, non-zero hours, because such a solution to the first-order conditions is not consistent with the constraint.[15]

5 Utility functions for applied use

This section discusses the implementation of our proposed preference class in applied contexts. This involves discussing the value of ν, special parametric forms, how the utility functions used in the literature relates to the utility functions here, along with some of their properties.

[15] For a very simple, static example, suppose $v(x)$ is a power function such that we can simply write $u(c, h) = -c^{-b}h^{d}$, where b and d are positive constants. In this case, u is strictly increasing in each of its arguments, and under a further assumption ($d > b + 1$) it is also strictly concave. One can show, for this case, that an interior choice for hours is only optimal if the consumer has negative wealth.

5.1 What is ν?

This question refers to the new element we add to standard preferences as represented by KPR's balanced-growth function (note that by setting $\nu = 0$ in Theorem 1 we obtain the KPR class in (1) above). The added element amounts to an enlargement of the KPR preference class and is captured by the parameter ν, which regulates the size of the additional income effect, i.e., that above and beyond the income effect present in the KPR class. With $\nu > 0$, hours fall when productivity rises as the income effect dominates the substitution effect. Similarly, if $\nu < 0$, hours rise when productivity rises. In the data, as we have seen in Section 2, hours fall as productivity rises, so the empirically relevant case is $\nu > 0$. In fact, we can use the time series data for an estimate of its value. Along a balanced growth path, productivity grows at (gross) rate γ, hours fall at rate $\gamma^{-\nu}$, and consumption as well as output grow at rate $\gamma^{1-\nu}$. Thus, with productivity and hours growing at, say, 2% and -0.4%, respectively (these numbers are rough averages), we would obtain an estimate of ν from $0.996 = 1.02^{-\nu}$. The result is a ν around 0.2. Thus, when restricting the analysis to long-run facts, the class of preferences we propose is not larger: it is different, thus necessitating a ν of 0.2 rather than $\nu = 0$.

The value of ν is key for the long-run behavior of hours, but of course the form of v can matter greatly for many other aspects of how much people choose to work: it matters for the level of hours (and therefore output and other macroeconomic aggregates), for transitional dynamics, for how hours respond to shocks of various kinds. We now look at a number of special cases.

5.2 Parametric forms

The applied utility function in the KPR class that we suggest was given by (3). The straightforward extension of this parametric class to $\nu \neq 0$ is obtained by simply writing, in place of h where it appears, $hc^{\frac{\nu}{1-\nu}}$. Thus we propose, as a rather flexible class for applied use,

$$u(c,h) = \frac{c^{1-\sigma}\left[1 - a\left(hc^{\frac{\nu}{1-\nu}}\right)^{b}\right]^{d} - 1}{1-\sigma}, \tag{13}$$

where we again have

$$a = \frac{\psi(1-\sigma)}{1+\frac{1}{\theta}}, \quad b = 1 + \frac{1}{\theta}, \quad \text{and} \quad d = (1-\sigma)\kappa.$$

Formally, this function is obtained by selection of a particular functional form for v: $v(x) = \left(1 - ax^{b}\right)^{\frac{d}{1-\sigma}}$ with $x \equiv hc^{\frac{\nu}{1-\nu}}$ in Theorem 1. This parametrization can straightforwardly be generalized further, e.g., by changing the functional form of the squared brackets of x (say, by adding an additional x term to some different power). However, as we show below, many interesting cases can indeed be viewed as special cases of (13). We thus view it as a natural starting point.

5.2.1 Generalized log-log

A particularly simple case is the generalization of the standard log-log case to $u(c,h) = \log(c) + \kappa \log\left(1 - \phi hc^{\frac{\nu}{1-\nu}}\right)$.[16] In Section 5.3.3 on transitional dynamics below, we will explore this case in some more detail for the special case $\kappa = 1$. As we will show there, the generalized log-log formulation we use here inherits the analytical convenience of the standard log-log case when combined with a Cobb-Douglas production technology and 100% depreciation.

[16] This case is obtained by setting $\psi = \phi/(1-\sigma)$ and $\theta \to \infty$ and letting $\sigma \to 1$.

5.2.2 A case of the Greenwood-Hercowitz-Huffman (1988) preferences

With $\kappa = 1$ and $\theta = -\nu$ in (13), where $\nu < 0$ and $\psi > 0$, we obtain the quasi-linear preferences

$$u(c,h) = \begin{cases} \frac{\left(c - \psi \frac{h^{1-\frac{1}{\nu}}}{1-\frac{1}{\nu}}\right)^{1-\sigma} - 1}{1-\sigma} & \text{if } \sigma \neq 1, \\ \log\left(c - \psi \frac{h^{1-\frac{1}{\nu}}}{1-\frac{1}{\nu}}\right) & \text{if } \sigma = 1. \end{cases} \tag{14}$$

This is an often used case of the Greenwood-Hercowitz-Huffman (1988; GHH for short) preferences.[17] In this class, the Frisch elasticity is constant and equal to $-\nu$. These preferences are non-homothetic but they are part of the Gorman class. Preferences (14) also imply a relative risk aversion in consumption, $-u_{cc}(c,h)c/u_c(c,h)$, which depends on $hc^{\frac{\nu}{1-\nu}}$. GHH preferences preclude any income effect on hours worked. Clearly, with a substitution effect alone, GHH preferences imply increasing hours as the wage rate increases. Consequently, we have $\nu < 0$ and there is no overlap with the KPR class. For concavity we need $\sigma > 0$. Note also that the preferences (14) fulfill Assumption 1 only with $1 > \psi \frac{x^{-\frac{1-\nu}{\nu}}}{1-\frac{1}{\nu}}$ and consequently additional restrictions are required.

Quasi-linear preferences are widely used in the applied theory and labor literatures, where the household problem is often assumed to be static and σ can be set to zero without loss of generality. However, the quasi-linear formulation does preclude income effects.

5.2.3 MaCurdy (1981)

With $\kappa = 1/(1-\sigma)$ and $\nu = \frac{\sigma-1}{\sigma+1/\theta}$ in (13), for $\sigma \neq 1$, we obtain the case considered by MaCurdy (1981) with

$$u(c,h) = \frac{c^{1-\sigma} - 1}{1-\sigma} - \psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}, \tag{15}$$

[17] A straighter path to this function is to set $v(x) = 1 - \psi \frac{x^{-\frac{1-\nu}{\nu}}}{1-\frac{1}{\nu}}$.

with $\sigma > 0$, $\psi > 0$, and $\theta > 0$.[18] The attractiveness of this functional form is that two important elasticities are controlled by two separate parameters: the intertemporal elasticity of substitution (IES) is constant and equal to $1/\sigma$ and the Frisch elasticity is equal to θ; we will discuss these two elasticities more broadly below. As is well known, with $\sigma \neq 1$, preferences of the form (15) are not part of the KPR class. For this reason, as already discussed in subsection 3.1, a significant part of the macroeconomic literature restricts itself to the case with a unitary IES by additionally setting $\sigma = 1$.[19] Then the preferences become $u(c,h) = \log(c) - \psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$ and are part of the KPR class.

Figure 10 below illustrates how σ and ν have to be restricted on a balanced path with falling hours: $\nu > 0$ requires $\sigma > \frac{1}{1-\nu} > 1$. Thus, any point on the downward-sloping curve is admissible (in the figure ν is set at a quantitatively reasonable value).

5.2.4 Departing from time invariance or time separability

Our preferences rely on there being a stationary utility function $u(c,h)$ characterizing choice. It is not altogether uncommon in the literature that people use utility functions that are either not stationary or not time-separable. As for non-stationarity, the typical assumption is that some elements of the period utility function shift with labor productivity: preferences change over time, in line with technology. Such functions are often motivated by (but not derived from) some form of home-production structure where the same productivity growth as in final-goods production occurs.[20]

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[18] One can also obtain this form directly by setting $v(x) = \left(1 + \frac{\psi\nu}{(1-\nu)} x^{\frac{(1-\nu)(\sigma-1)}{\nu}}\right)^{\frac{1}{1-\sigma}}$ for $\sigma \neq 1$.

[19] For instance, Shimer (2010) proposes this preference specification in chapter 1 of his textbook and then writes "This formulation imposes that preferences are additively separable over time and across states of the world. It also imposes that preferences are consistent with balanced growth—doubling a household's initial assets and its income in every state of the world doubles its consumption but does not affect its labor supply. [...] I maintain both of these assumptions throughout this book."

[20] For a model of structural change between home and market production that generates long-run changes in hours worked in the market place, see Ngai and Pissarides (2008).

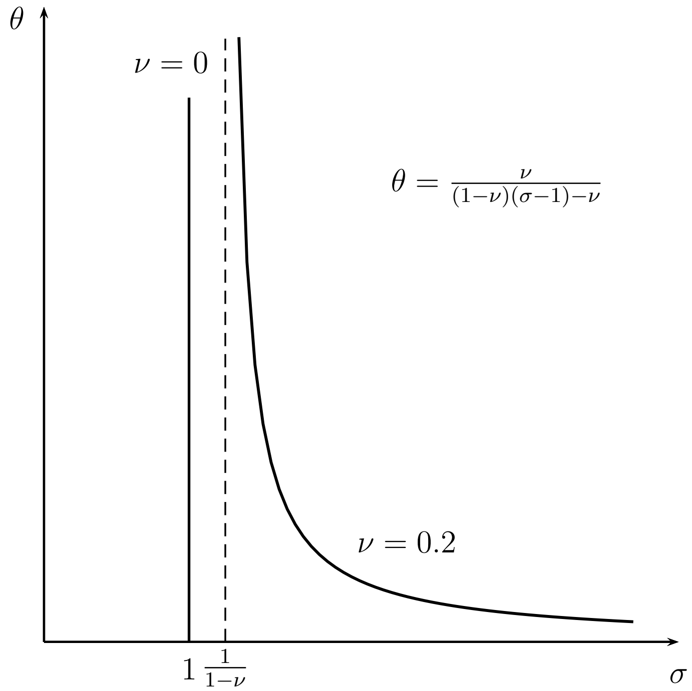

Figure 10: Combinations of elasticities

The figure shows combinations of relative risk aversion in consumption σ and Frisch elasticity θ in the functional form (15) that are consistent with (i) constant hours ($\nu = 0$) and (ii) hours falling at rate $\gamma^{-0.2}$. With two percent productivity growth, i.e., $\gamma = 1.02$, and $\nu = 0.2$ hours worked decline at roughly 0.4 percent per year.

As an example, one can make (15) consistent with constant hours in the long run by adding a time-varying term in front of ψ that is growing at the appropriate rate (see, e.g., Mertens and Ravn, 2011). Such a formulation has been deemed useful when one wants to consider free curvature in consumption and hours separately and yet not violate the balanced-growth conditions. Similarly, by adding a term that grows at the rate of technical change in front ψ, also the GHH case in (14) can be made consistent with constant hours worked. A deeper foundation for this kind of utility function is proposed by Hercowitz and Sampson (1991), who appeal to human capital accumulation—which, from a home-production perspective, would raise the disutility of work.

Reconciling constant hours worked in the long run with a small (or non-existent) income effect has also attracted some attention in the macroeconomic literature, since small income effects are sometimes appealing when it comes to fluctuations around a balanced growth path. Hence, the literature has extended the KPR class by giving up the assumption of time separability. A particularly well-known case is Jaimovich and Rebelo (2009). Our analysis shows that even the GHH utility function (14) is part of the general balanced-growth preferences specified in Theorem 1 but of course, as discussed above, they would imply increasing hours worked as wages grow. However, by adding a "habit" term $X_t = c_t^{\rho} X_{t-1}^{1-\rho}$ in front of $h^{1-\frac{1}{\nu}}$ to these preferences, we can also obtain the preferences studied in Jaimovich and Rebelo (2009).

The purpose here is not to take issue with preference formulations that depart from time invariance or time separability. Relatedly, whether the additional terms in such formulations ought to be exogenous to the individual (external) or controlled by the individual (internal) is not a question we address here. It suffices to say that the "tricks" that have been employed in the literature are still possible to employ under our preference class. We will return briefly to the possibility of introducing externalities in Section 6.2.

5.3 Often-discussed features of utility

Within the proposed parametric class (13) there are many possible functions that are all consistent with balanced growth under falling hours but which differ in important properties. Some of these properties are now discussed. We begin with curvature, then turn to the Frisch elasticity of labor supply, and finally discuss qualitatively different transitional dynamics in two special cases of the model where we also make assumptions on technology for sake of tractability.

5.3.1 Consumption curvature

The IES—the intertemporal elasticity of substitution of consumption—measures a form of curvature in consumption and is a key object in some macroeconomic analyses. In a time-additive setting without an hours choice, it is also one divided by the coefficient of relative risk aversion, $-u''(c)c/u'(c)$. However, in a context where leisure is valued, it is more difficult to define these concepts. A natural measure of risk aversion would define a lottery over consumption and hours, or over wealth; Swanson (2015) discusses this question in detail. Thus, what we defined as our RRA_c function above, $-u_{cc}(c,h)c/u_c(c,h)$, is not the most natural measure of risk aversion: it is defined as lotteries over consumption only, keeping h fixed. Turning to its characterization, Lemma 2 shows that it is endogenous—it is a function of $hc^{\frac{\nu}{1-\nu}}$—but it is constant along the balanced growth path.

Similarly, the definition of the IES, i.e., $d\log(c_{t+1}/c_t)/d\log(1+r)$, where r is the net interest rate between t and $t+1$, is more complicated when the utility function includes hours worked. Along the lines of the definition above, one can define a restricted IES notion keeping h_t and h_{t+1} constant. When evaluated on the balanced growth path one then obtains the IES $1/(-u_{cc}(c,h)c/u_c(c,h))$, which again is constant over time from Lemma 2, but a function of $hc^{\frac{\nu}{1-\nu}}$. Thus, one obtains one divided by the curvature measure used above for the restricted notion of the RRA_c.

Here we first wish to re-emphasize the point just made: although the relative risk aversion in consumption remains constant on a balanced growth path, it can be endogenously determined. In contrast, in the standard KPR setting, the IES is not only constant on a balanced growth path but exogenous. We have the following.

Proposition 1. *Given the preferences specified in Theorem 1, with $\nu = 0$, the intertemporal elasticity of substitution is independent of c and h: it equals $1/\sigma$. With $\nu \neq 0$, however, the intertemporal elasticity of substitution can, but will not necessarily, depend on $hc^{\frac{\nu}{1-\nu}}$.*

Proof. For the KPR class this is verified straightforwardly. For the case $\nu \neq 0$, two cases are dealt with in the text below: one where the IES is decreasing in $hc^{\frac{\nu}{1-\nu}}$ and one where it is constant and exogenous (and equal to $1/\sigma$). ■

For the MaCurdy formulation, it is straightforwardly verified that the $\mathrm{RRA_c}$ is exogenous and equals σ. However, under GHH, it is equally straightforwardly verified, that the $\mathrm{RRA_c}$/IES is indeed endogenous. For many applications, perhaps particularly in asset pricing, it may be interesting to consider preferences where the $\mathrm{RRA_c}$ is decreasing in the consumption-hours aggregate $hc^{\frac{\nu}{1-\nu}}$: in this case, booms involve lower consumption curvature. One can imagine different functional-form assumptions here but one is yet another special case of (13). Thus, first set $\kappa = 1/(1-\sigma)$ and we obtain

$$u(c,h) = \frac{c^{1-\sigma}-1}{1-\sigma} - \psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}} c^{\frac{\nu}{1-\nu}(1+\frac{1}{\theta})+1-\sigma}. \tag{16}$$

This, clearly, is a slight extension of the MaCurdy case, allowing a variable $\mathrm{RRA_c}$. Further, set $1+\frac{1}{\theta} = \epsilon$ and $\frac{1-\nu}{\nu} = \epsilon$. We then obtain the functional form

$$u(c,h) = \frac{c^{1-\sigma}}{1-\sigma} - \psi \frac{h^{\epsilon} c^{2-\sigma}}{\epsilon}, \tag{17}$$

for $\psi > 0$, $\sigma > 2$ and $\epsilon > \sigma - 1$. In this case, we have

$$\mathrm{RRA_c} = \sigma - \frac{(\sigma-2)\psi\frac{h^{\epsilon}c}{\epsilon}}{1+(\sigma-2)\psi\frac{h^{\epsilon}c}{\epsilon}}, \tag{18}$$

which is decreasing in $x = hc^{\frac{\nu}{1-\nu}} = hc^{\frac{1}{\epsilon}}$.

5.3.2 The Frisch elasticity

The Frisch elasticity of labor supply is defined as the percentage change in hours when the wage rate is changed by 1 percent, keeping the marginal utility of consumption constant. It is a useful concept in the context of intertemporal substitution of hours worked when there is a frictionless market for borrowing and lending: it relies on the notion that whatever extra labor income is earned by working harder is in part substituted toward other periods. This is captured by the requirement that the marginal utility of consumption remain unchanged; it would change if the income had to be consumed today.

The functional form we propose in Theorem 1 implies that the Frisch elasticity is constant along the balanced growth path. To see this, note that we can write $u_2(c,h) = \frac{c^{1-\sigma}}{h}z_1(x)$, $u_{11}(c,h) = c^{-\sigma-1}z_2(x)$, $u_{22}(c,h) = \frac{c^{1-\sigma}}{h^2}z_3(x)$, and $u_{12}(c,h) = \frac{c^{-\sigma}}{h}z_4(x)$, where z_1, z_2, z_3, and z_4 are some functions of $x = hc^{\frac{\nu}{1-\nu}}$. Inserting these expressions into the formula for the Frisch elasticity, $\frac{u_2(c,h)u_{11}(c,h)}{h[u_{22}(c,h)u_{11}(c,h)-u_{12}(c,h)^2]}$, one immediately sees that the Frisch elasticity only depends on x, which remains constant along a balanced growth path. Note, however, that its balanced-growth value will be endogenous and will in general depend on model parameters; in a model with heterogeneous agents it would also differ across rich and poor, and so on.

For the KPR formulation, as pointed out below, Trabandt and Uhlig (2011) provide a theorem specifying for which subclass of KPR that the Frisch elasticity is constant, i.e., independent of (c,h): it is constant under $u(c,h) = \log(c) - \psi\frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$ and it is constant under $u(c,h) = \frac{c^{1-\sigma}}{1-\sigma}\left(1 - \kappa(1-\sigma)\frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}\right)^{\sigma}$ (for $\sigma \neq 1$ and $\kappa > 0$), but it is constant for no other function.[21] Our formulation, for $\nu \neq 0$, appears harder to characterize fully in this regard, but it is clear that there are two additional cases in which the Frisch elasticity is constant. One is our GHH formulation, for which the Frisch elasticity is equal to $-\nu$. Interestingly, although the Frisch elasticity for any given function $u(c,h)$ generally does not feature invariance with respect to monotone transformations—the elasticity for u is different than that for $f(u)$, where f is monotone—it is actually invariant in the GHH case. The second case obtains with the MaCurdy function. So far, we know of no other cases than these two.

5.3.3 Transitional dynamics: two simple examples

Under some additional conditions, as discussed in Section 4.3.5 above, the preferences in Theorem 1 guarantee the existence of a balanced growth path for any neoclassical production function. Along this balanced growth path, the growth rates of all the variables are completely characterized by the rate of technical change γ, population growth η, as well as our new preference parameter ν. For transition dynamics, it is useful to detrend any growing (or shrinking) variables with their respective growth rates. Detrended capital—the only state variable in our one-sector model—is $\hat{k}_t \equiv \frac{K_t}{\eta^t \gamma^{(1-\nu)t}}$, whereas we define detrended hours worked and consumption as $\hat{h}_t \equiv \frac{h_t}{\gamma^{-\nu t}}$ and $\hat{c}_t \equiv \frac{c_t}{\gamma^{(1-\nu)t}}$, respectively. Thus, on the balanced path, $\hat{k}$ is constant and this value, as usual, is determined jointly by the Euler equation, which delivers $\hat{k}/\hat{h}$ as a function of preference and technology parameters, and the first-order condition for hours, which involves $\hat{k}$ and $\hat{h}$, where for each equation the resource constraint has been used to eliminate the consumption variable. For given model parameters the transitional dynamics can then be solved for numerically using standard methods, e.g., using linearization around the steady-state value $\hat{k}$. It

[21] Recall that we nest the former as a special case of the latter with our formulation of a parameterized KPR family in (3).

is beyond the scope of the present analysis to fully explore how the preferences we propose alter transition dynamics in well-known models. Rather, we will illustrate different possibilities with two simple examples where the dynamics can be solved for in closed form. In the first case, we will focus on preferences with $\sigma = 1$ and in the second, we will examine the MaCurdy class. As we shall see, the transitional dynamics for hours are qualitatively different in these two cases. Both cases involve 100% depreciation of capital, which does not appear restrictive to us since the focus here is on growth and the time period can be chosen to be long enough that such an assumption is not so unrealistic.

Cobb-Douglas production with $\sigma = 1$ So let us assume that $\sigma = 1$, that the production function is Cobb-Douglas, i.e., $K_t^{\alpha}\left(\gamma^t h_t \eta^t\right)^{1-\alpha}$, and that there is 100% depreciation. Under these assumptions the first-order conditions for the planner's solution can be expressed (in terms of detrended variables) as

$$\eta\gamma^{1-\nu}\hat{k}_{t+1} = \hat{k}_t^{\alpha}\hat{h}_t^{1-\alpha} - \hat{c}_t,$$

$$-(1-\alpha)\left(\frac{\hat{k}_t}{\hat{h}_t}\right)^{\alpha} = \frac{\hat{c}_t}{\hat{h}_t}\left[\frac{\frac{v'(x_t)}{v(x_t)}x_t}{1+\frac{\nu}{1-\nu}\frac{v'(x_t)}{v(x_t)}x_t}\right],$$

$$\frac{\hat{c}_{t+1}}{\hat{c}_t}\frac{1+\frac{\nu}{1-\nu}\frac{v'(x_t)}{v(x_t)}x_t}{1+\frac{\nu}{1-\nu}\frac{v'(x_{t+1})}{v(x_{t+1})}x_{t+1}} = \frac{\beta\alpha}{\gamma^{1-\nu}}\left(\frac{\hat{h}_{t+1}}{\hat{k}_{t+1}}\right)^{1-\alpha}.$$

It is straightforward to guess, and verify, that a constant saving rate of $\alpha\beta\eta$, i.e., $\hat{c}_t = (1-\alpha\beta\eta)\hat{k}_t^{\alpha}\hat{h}_t^{1-\alpha}$, a constant $x_t \equiv \bar{x}$, and hours supply of $\hat{h}_t = \left(\frac{\bar{x}^{1-\nu}}{(1-\alpha\beta\eta)^{\nu}}\hat{k}_t^{-\nu\alpha}\right)^{\frac{1}{1-\nu\alpha}}$, satisfies these conditions with

$$-\frac{1-\alpha}{1-\alpha\beta\eta}\left[v\left(\bar{x}\right)+\frac{\nu}{1-\nu}v'(\bar{x})\bar{x}\right] = v'\left(\bar{x}\right)\bar{x} \tag{19}$$

determining the solution for $\bar{x}$.[22] The speed of convergence $-\frac{\partial \log\left(\hat{k}_{t+1}/\hat{k}_t\right)}{\partial \log \hat{k}_t}$ is given by $\frac{1-\alpha}{1-\alpha\nu}$ and consequently strictly increasing in ν. Obviously, not any function v can be used here, and the Appendix B.2 provides an example with a simple functional form for $v(x)$ such that (19) can be solved explicitly for $\bar{x}$. Since the Frisch elasticity can be written as a function of x_t alone the closed-form solution also implies a constant Frisch elasticity even along the transition.

The solution for hours just derived implies that if the initial capital stock is below its balanced-growth level, hours worked will be above their balanced-growth level. This illustration shows how transitional dynamics might be helpful for understanding the high and steeply decreasing hours worked in France or Germany after World War II resulting from a relatively low physical capital stock. Of course, the precise shape of the transitional dynamics depend on the functional form of preferences and technology, so the example here is only meant as an illustration and the dynamics do not generalize; for example, below we construct a case where x adjusts over the transition but where detrended hours are constant.

MaCurdy preferences with CES production Suppose preferences are $\frac{c^{1-\sigma}-1}{1-\sigma} - \psi\frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$, with $\sigma \neq 1$, and again let us focus on a planning problem. Here, a Cobb-Douglas production function does not admit a closed-form solution. However, one can show that the result in Koulovatianos and Mirman (2007) can be extended to the MaCurdy case (and thus with endogenous hours): a closed-form solution with a constant saving rate along the transition path obtains when $\sigma/\varepsilon = 1$, where ε is the degree of substitutability between capital and hours in a CES production function. Thus, a constant saving rate can be optimal for very high consumption curvature and hence a *low* intertemporal consumption substitutability so long as the production function features a correspondingly *high* elasticity of transformation

[22] The closed-form solution also obtains in the presence of TFP variations, predictable or not. Hours then respond to TFP but x and the saving rate are constant.

across time. The case where $\sigma = \varepsilon = 1$ is well known here but the other cases appear less well known.

Given that we can extend the result to to the MaCurdy preference class, how are hours chosen in this case? It is straightforward to show, as is detailed in Appendix B.2, that detrended hours, $\hat{h}$, are constant during transition. That is, unlike in the case we looked at first, where the hours-consumption composite $hc^{\frac{\nu}{1-\nu}}$ is constant and hours are higher when the economy starts out with low capital, here detrended hours remain constant whereas $hc^{\frac{\nu}{1-\nu}}$ is moving monotonically. The saving rate out of gross output remains constant along the transition and is equal to $1 - \eta(\alpha\beta)^{\frac{1}{\sigma}}$. Interestingly, with the CES production function factor shares change along the transition.[23]

6 Consumer heterogeneity and cross-sectional facts

We now comment briefly on two important concerns. One is that the analysis so far has exclusively looked at a representative-agent economy, so a natural question is whether our results are robust to introducing consumer heterogeneity of various forms. Another concern has to do with the cross-sectional implications of our preference class: if income effects exceed substitution effects, it seems that high-wage workers would work less than low-wage workers and perhaps make different portfolio choices too. We therefore also discuss this aspect below.

6.1 Models of consumer heterogeneity

Our theory of labor supply in the long run, strictly speaking, holds only for a representative-agent economy. Is it relevant, then, in cases when aggregation does

[23] The closed-form solution with a constant saving rate also extends to a case where there are additional time-varying (random or deterministic) Hicks- or Harrod-neutral technology terms. Harrod-neutral technology movements then cause $\hat{h}$ to deviate from its balanced-growth value, whereas Hicks-neutral technology movements do not.

not hold? Whereas it is beyond the scope of the present paper to provide a full answer to this question, let us still conjecture in the affirmative. More precisely, we conjecture that in an environment with a stationary distribution of agents heterogeneous in assets, wages, utility-function parameters, etc., preferences in our class are needed to match the aggregate growth facts (including aggregate hours shrinking at a constant rate).

The reason for this conjecture is perhaps best explained with an example. So consider the modern macro-style models of inequality: the Bewley-Huggett-Aiyagari model. By now, this model has been extended and used in a vast variety of applications, with the common element being that there are incomplete markets for household-specific idiosyncratic shocks of different kinds and implied differences in wealth and consumption. Many of these models also consider substantial additional heterogeneity, such as in preferences (see, e.g., the multiple-discount factor model in Krusell and Smith, 1998), and yet others consider life-cycle versions with and without bequest motives (see, e.g., Huggett, 1996).

These modern-macro models of inequality, then, do not display aggregation (at the very least due to incomplete markets) and they are typically analyzed in steady state. A key question, thus, is: for standard KPR preferences, and more generally for the broader class of preferences considered here, do these models admit balanced growth? The answer is yes. It is straightforward to transform variables and verify this assertion, just like a representative-agent model would be rendered stationary by variable transformation. Of course, growth makes a difference—the discount rate(s), for example, would need to be transformed—so that some aspects of the aggregate variables (such as the capital-output ratio) will depend on the rate of growth, as will the moments of the stationary distribution of wealth. In Appendix B.1, we formally prove this assertion for a typical Aiyagari model with preference heterogeneity in discounting (and it should be clear from the analysis there that many other sources of heterogeneity could be handled as well). Why does the transformation of variables work? It is straightforward to see that it is precisely because the preferences are in the pre-specified class, and for this reason we conjecture that it would not work outside of this class: balanced growth is, by definition, a set of paths for the economy's different variables that can be rendered constant by a standard transformation.

Can life-cycle models with shrinking hours be accommodated? Though the model in Appendix B.1 has infinitely-lived agents, it should be clear from the analysis there that the answer is yes. However, in the context of life-cycle models, one is also led to think about participation, and perhaps how participation changes as life expectancy and productivity rises. The modeling above focuses entirely on the intensive margin; what can then be said from a theoretical perspective on the extensive margin and how it reacts to productivity growth? How does the extensive margin depend on the form of the utility function? We have begun thinking about these issues, but a full study must be conducted as a separate project. Let us therefore just make a few observations. So suppose that hours are restricted to be either 0 or a positive, exogenous amount $\bar{h}$ (say, per month or week). Then, first of all, some features of the utility function will not matter. To illustrate, consider the MaCurdy function. Clearly, what matters for choice, then, is the difference between 0 and $\psi\frac{\bar{h}^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$. Hence, the curvature (which contains ν, a key parameter in this paper) is unimportant per se. Of course, if there are two possible choices for positive hours (say, half-time and full-time), then curvature again becomes important. Second, suppose one wants to derive the prediction that the participation rate decreases over time: would this be possible for the utility function class we propose? In general, this may be difficult, but suppose one imagines a planner allocating work to households in the population, 0 or $\bar{h}$, and suppose labor productivity is growing. Then if $u(c,h)$ is not additively separable, the efficient allocation involves different consumption levels across working statuses. Moreover, in this case, it appears difficult to obtain a balanced-growth equilibrium where the participation rate shrinks at a constant rate. However, if $u(c, h)$ is additively separable—the MaCurdy case—it is, in fact possible, so long as $\sigma > 1$. It is thus not possible with KPR preferences, which require $\sigma = 1$. Having said all this, a more realistic model would have decentralized markets with less than full insurance, etc., and the implications of such a model are less obvious. One would ideally construct such a model with both an intensive and an extensive margin.

6.2 Cross-sectional data

Another concern one might have from a perspective of heterogeneity is that the model with an income effect that is larger than the substitution effect might be inconsistent with what we know from cross-sectional data on households. In particular, there seems to be a view that consumers with higher wages work more, and not less, as would be implied by our theory. Regarding portfolio choice, there are potentially also implications for cross-sectional data that differ from those derived under KPR preferences. We now make a sequence of remarks on this issue. None of these remarks settles the issue entirely and, as in the case with the participation margin, a fuller analysis is needed and such an analysis is outside the scope of the present paper.

First, we are not entirely sure of what the data says. Ideally, one would want a life-time, all-inclusive hours measure and then ceteris-paribus experiments where a permanent wage is changed across households. Arguably, convincing such studies are hard to come by. Interestingly, there is in fact a recently published study that claims that the wage-hours correlation is negative, not positive. In particular, the study of the intensive margin in Heathcote, Storesletten, and Violante (2014) reports such a correlation, after taking out time dummies and age effects. We note in this context that based on the consumer data they look at, Heathcote, Storesletten and Violante (2014) actually use a MaCurdy preference formulation—thus, one in our class—that implies an income effect that exceeds the substitution effect. In fact, with their estimates that implies $\nu = 0.184 > 0$ and annual wage growth of 2 percent, the implied annual growth rate of hours worked over time is -0.365 percent, so quite in line with the international/historical data. As for what the data says about the extensive margin, it is well documented that the highly educated work longer, but they also start working later. Thus, how life-time participation varies with measures of permanent wage is also not clear.

If, however, the perception that high-skilled people work more, not less, is correct, then we must point out that such a fact would be difficult to explain also with the standard model, i.e., with KPR preferences: our generalization would merely make the challenge slightly more difficult. There are studies in the literature that have attempted to address this issue, using a combination of assumptions. One is that the high wages that are observed—and are observed to be associated with higher working hours—represent a temporary window of opportunity. For such a situation, our preference class is consistent with a positive correlation. Another possibility is a non-convexity of the budget set of consumers in the form of a wage rate that depends on the amount of hours worked (see Erosa, Fuester and Kambourov, 2015). One can easily imagine other channels. Suppose, for example, that people differ in their "utility cost of effort".[24] Then those with high costs will work less and presumably, when effort toward education and learning is factored in, also obtain lower wages. Thus, a positive correlation between wage and hours would be generated in the cross-section. Other elements of heterogeneity could, it seems, also deliver the same qualitative result and it is an open question what amount of heterogeneity would be necessary to turn a negative into a positive hours-wage correlation in the cross-section.

Another avenue for explaining the cross-section is to short-circuit the income

[24] A formulation of such heterogeneity is entertained in Bick et al. (2015), with a utilitarian planner choosing who works and who does not at any point in time. This formulation would lead, in reduced form, to a utility function of the kind $\log c - m(h)$, where c is per-capita consumption, equalized across all agents, and h is total employment. Bick et al. combine this setting with a Stone-Geary element in order to obtain decreasing hours during the transition.

effect on the individual level but maintain it on the aggregate level, through an externality. This approach would be rather ad hoc but it would work as follows: replace the c in $hc^{\frac{\nu}{1-\nu}}$ in our setting with $c^{\xi}\bar{c}^{1-\xi}$, where $\bar{c}$ is the average consumption level in the economy. Then, if $\xi = 0$, on the individual level, the income effect would equal the substitution effect. One can further weaken the income effect by considering a $\xi < 0$, thus producing a positive hours-wage correlation in the cross-section. Thus, taken together, there are many ways to generate a positive wage-hours correlation, some explored in the literature and some not, and it remains to address the balanced-growth facts with these theories. We believe that a full evaluation of the cross-sectional aspects of the model requires much more work and that it is far too early to dismiss KPR preferences (or those in our proposed class).

Regarding portfolio choice, a stylized fact appears to be that rich households hold a larger share of their wealth in risky assets. This phenomenon does not allow itself to be straightforwardly explained by a heterogeneous-agent version—thus extended to include risk—of theory based on KPR preferences. For example, a theory based on non-homothetic preferences is proposed by Wachter and Yogo (2010). Can the preferences entertained in this paper help generate cross-sectional implications that are in line with the data? To answer this question appears to be a research project in itself but it is clear that at least consumption curvature will be endogenous in this model and potentially decreasing in a wealth index, as discussed above. However, the relevant risk-aversion measure in this model must be derived from the value function, which is a function in wealth/cash on hand. We hope to explore attitudes to risk from the perspective of the preferences proposed here in future work.

7 Hours worked in the postwar U.S.

Our data section emphasized the postwar U.S. experience with its rather stationary hours as an exception rather than a rule, in an international and historical comparison. Equipped with our theory, what can we say about possible explanations for the exceptional behavior of the U.S. data? The purpose of the short present section is not to provide an ambitious and full account of the observations but rather to suggest some possibilities. In addition, there are a number of existing related studies. For example, there is a clear connection with the debate on the comparison between hours worked in Europe and in the U.S. initiated by Prescott (2004) and Rogerson (2006, 2008). These authors argue that the gap in hours worked that opened up over the postwar period reflects relative changes in tax rates: hours worked fell in Europe (in relative terms) because of upward movements of tax rates in Europe relative to those in the U.S.[25]

To contrast the U.S. experience with some of the more extreme European cases, consider Figure 11 which displays the U.S. data on hours together with that for Germany and France. Clearly, hours fell at a fast rate in these European economies, indeed at a much faster rate than in the broader cross-section of countries we looked at in the data section of the paper: the cross-country average is a rate of decline in hours that lies in between that of the U.S. and those in Germany and France.

Turning to possible explanations for the U.S. experience, let us comment on the Prescott-Rogerson argument first. Here, our model only calls for a slightly different interpretation of the data. Whether income effects are slightly larger than substitution effects or not, higher taxes and a larger transfer system would lower hours worked and so if these policy variables diverged between two regions, one would expect hours worked to diverge too. Moreover, since we do not argue for income effects being much higher than the substitution effects, the quantitative effects on the relative paths of hours in Europe and the U.S. would probably be similar to those computed by Prescott and Rogerson. The only difference is that

[25] Interestingly, Ohanian, Raffo, and Rogerson (2008), look at the developments in a number of OECD countries, along with tax-rate data, and also point to falling hours. They interpret this phenomenon as transitional and model it with Stone-Geary preferences as in the approach taken by Bick et al. (2015).

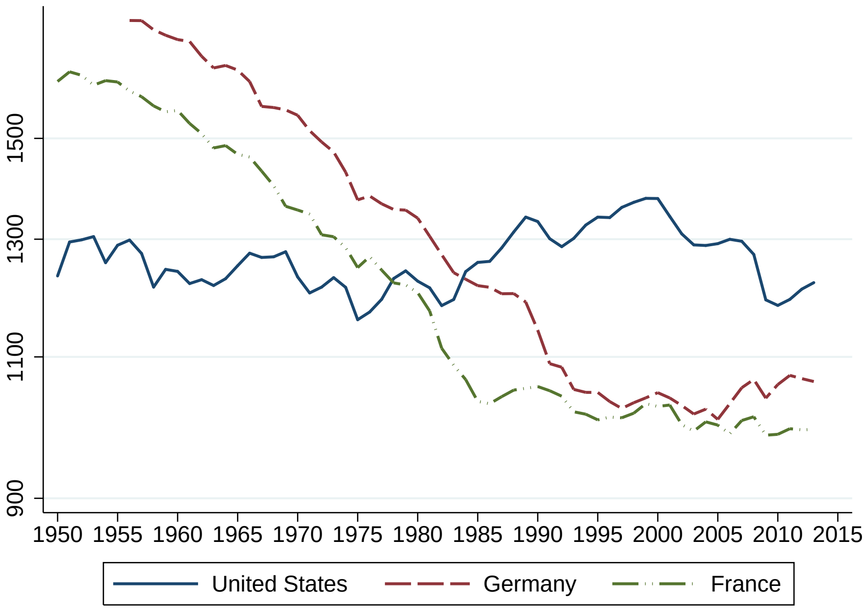

Figure 11: **Hours worked in the U.S., Germany, and France, 1950–2013**

Notes: Source: GGDC Total Economy Database for total hours worked and OECD for the data on population aged 15–64. The figure is comparable to the ones in Rogerson (2006).

our perspective would suggest that stable hours, while productivity kept growing, would in that case have required a fall in taxes and transfers. Indeed, the tax cuts of the Reagan years represent such a phenomenon. Moreover, prior to the Reagan tax cuts U.S. hours seemed to be falling, so the timing appears roughly consistent with this argument.

Second, there has been demographic change, with the baby boom standing out as a major factor. Figure C.4 in the appendix shows the implied movements of aggregate U.S. hours due to demographics based on a mechanical view of labor supply by age (using current hours per adult by age to project backwards). Clearly, demographics account for a fall in labor supply after the 1950s and then a turnaround only in the second half of the 1970s, and thus can be an important factor to take into account.

Third, another potential factor comes from the observation that median wages have not grown much at all in the U.S., and per-capita hours are un-weighted by productivity/wages. That is, if the vast majority of the population does not experience wage growth, constant hours is of course consistent with our theory. Thus, the well-documented increase in wage inequality from the late 1970s and onwards—which coincides with the years within the postwar period with rising hours worked—is a possible third factor.

A fourth factor is women's increased labor-force participation. This feature of the data may be explained by some form of discrimination, where women then would formerly have been constrained and not able to work (at appropriate wages). Another perspective is that technical change in household production lies behind the change in female labor supply. Either way, aside from studying its roots, one would need to simultaneously address the slight downward trend in male participation. Overall, however, the participation dynamics at least mechanically account for some of the changes in the aggregate hours data.

It is beyond the scope of the present paper to evaluate these different potential explanations quantitatively; in fact, in a separate research project we have indeed begun to address the exceptional behavior of the postwar U.S. hours data using a structural model incorporating all these factors. Going forward, what should one expect about U.S. labor supply (and that in other countries)? Our theory simply says that, ceteris paribus, hours will fall at a modest rate to the extent the historical record on growth in labor productivity continues. Assuming that there will be continued productivity growth, then, could we expect counteracting factors based on the discussion above? Demographic change will certainly play some role but changes in the retirement behavior as a function of the projected increases in life expectancy would need to be taken into account as well. The inequality trend could continue, thus allowing average productivity growth but no median productivity growth. Participation could change further, but there is a clear limit to the increased participation rate of women: this is a clear transitional phenomenon by definition.

Finally, for taxes to provide a balance against productivity growth, they would have to keep falling.

8 Conclusions

We have presented an extension to the standard preference framework used to account for the balanced-growth facts. The new preference class admits that hours worked fall at a constant rate when labor productivity grows at a constant rate, as we have also documented the data to show across history and space. The new preference class intuitively involves an income effect of productivity on hours worked that exceeds the substitution effect.

We believe that our new preference class has potentially interesting implications in a range of contexts. As for growth theory and growth empirics, note that on our balanced path, the main macroeconomic aggregates (output, investment, consumption) grow at the rate $\gamma^{1-\nu} > 1$ (ignoring population growth), i.e., at a rate lower than productivity and in a way that is determined by the preference parameter ν. Notice also that from a development perspective, falling hours worked is not a sign of economic malfunctioning but rather the opposite: it is the natural outcome given preferences and productivity growth, and it rather instead illustrates clearly how output is an incomplete measure of welfare (see Jones and Klenow, 2015): leisure grows. Interestingly, our theory says that growth theory probably should not abstract from labor supply (which is typically set to "1" in models); rather, it seems an important variable to model as it determines the growth of long-run output in conjunction with the process of technical change.

Does our preference class have something to say about business-cycle analysis? We cannot identify any immediate substantive implications, but it is clear that our model can be amended with shocks and transformed to a stationary one that can be analyzed just like in the RBC and NK literatures. The preference class consistent with hours falling at a constant, but low, rate is a bit different than the standard one. From the perspective of a particular case—the MaCurdy constant-Frisch elasticity functional form—one can admit an arbitrarily low elasticity of intertemporal substitution of consumption, though only if the Frisch elasticity is then also very low.

Other areas where the new preference class may be interesting to entertain include asset pricing and public finance. For asset pricing—as we showed in the paper—it is possible to have attitudes toward risk behave qualitatively differently, and possibly more in line with data, than using standard balanced-growth preferences. These same features would potentially also help explain portfolio-choice patterns across wealth groups. For public finance, the sustainability of government programs, such as social security, and debt service in the future depend greatly on how hours worked will develop (along, of course, with the development of productivity).

We build the explanation for the secular drop in hours into preferences. What about institutional factors? We take the view that over a long horizon, they must be endogenous and thus must be responding to preferences. Moreover, the facts—an approximately constant rate of decline in hours worked—are too stark not to propose a "deep", and time- and space-independent, explanation. Of course, we are open to alternatives but our approach seems a reasonable place to start. What are, then, alternative theories that could explain why hours fall? Could an alternative theory explain the past without contradicting the constant-hours presumption of the standard macroeconomic model? Other mechanisms for income effects dominating substitution effects are possible, such as the Stone-Geary formulation proposed in Bick et al. (2015), following Atkeson and Ogaki's estimates (1996). Whether the transition dynamics in such a model are slow enough to generate the long-term, constant percent decline in hours observed in the data we look at here is an open question.

In sum, our present analysis should be viewed as one way to look at the long-run data, and it should carefully be compared to others, especially since their implications for the future differ markedly. For example, the Stone-Geary formulation implies that the future will see flat hours, independently of how future productivity evolves, whereas the implications of the preferences we propose here suggest a tight hours-productivity link. Our theory also has a number of other implications (over the business cycle, for asset-pricing, growth, and so on) and suggests avenues for follow-up research. We hope to address some of these applications in future work.

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Appendix A

A.1 Proofs

We now present the proofs of Lemma 1, Lemma 2 and Theorem 1.

We start by proving the two lemmata, characterizing the marginal rate of substitution (MRS) function between c and h and characterizing the curvature with respect to consumption: the relative risk aversion in consumption (RRA_c) function. The proof of Theorem 1 then uses these lemmata to derive the final characterization. Because the proofs will involve a large number of auxiliary functions that are either functions of $hc^{\frac{\nu}{1-\nu}}$ or of h, we economize somewhat on notation by sometimes denoting $hc^{\frac{\nu}{1-\nu}}$ by x and by systematically letting f_i be a function of x whereas m_j is a function of h (where i and j are indices for the different functions we will define). A sequence of constants will also appear; they are denoted A_k, accordingly, from $k = 1$ and on.

Proof of Lemma 1

Proof. Because λ is arbitrary, we can set it in (10) so that $c\lambda^{1-\nu} = 1$. This delivers

$$-\frac{u_2(1, hc^{\frac{\nu}{1-\nu}})}{u_1(1, hc^{\frac{\nu}{1-\nu}})} = wc^{-\frac{1}{1-\nu}}.$$

Evaluating (10) at $\lambda = 1$ we obtain $-\frac{u_2(c,h)}{u_1(c,h)} = w$. Inserting this expression, we thus obtain

$$\frac{u_2(c, h)}{u_1(c, h)} = c^{\frac{1}{1-\nu}} \frac{u_2(1, hc^{\frac{\nu}{1-\nu}})}{u_1(1, hc^{\frac{\nu}{1-\nu}})}. \tag{A.1}$$

Now identifying $q(x)$ as $\frac{u_2(1,x)}{u_1(1,x)}$, where $x = hc^{\frac{\nu}{1-\nu}}$, gives the result in Lemma 1. ∎

It follows from Lemma 1 and u being twice continuously differentiable that q is continuously differentiable.

Proof of Lemma 2

Proof. The second first-order condition, (11), holds for all λ so it can be differentiated with respect to λ and then evaluated at $\lambda = 1$ and divide by (11) again to yield

$$(1-\nu)c\gamma^{1-\nu}\frac{u_{11}\left(c\gamma^{1-\nu}, h\gamma^{-\nu}\right)}{u_1\left(c\gamma^{1-\nu}, h\gamma^{-\nu}\right)} - \nu h\gamma^{-\nu}\frac{u_{12}\left(c\gamma^{1-\nu}, h\gamma^{-\nu}\right)}{u_1\left(c\gamma^{1-\nu}, h\gamma^{-\nu}\right)} = (1-\nu)c\frac{u_{11}\left(c, h\right)}{u_1\left(c, h\right)} - \nu h\frac{u_{12}\left(c, h\right)}{u_1\left(c, h\right)}. \tag{A.2}$$

This equation has to hold for all γ (and consequently one must adjust R, but R does not appear in the equation). Moreover, it has to hold for all c and h; it has to hold for all h because Assumption 1 allows any w and hence any h (given an arbitrary c). Given this, by setting γ so that $c\gamma^{1-\nu} = 1$ we can state (A.2) as

$$(1-\nu)\frac{u_{11}(1, hc^{\frac{\nu}{1-\nu}})}{u_1(1, hc^{\frac{\nu}{1-\nu}})} - \nu hc^{\frac{\nu}{1-\nu}}\frac{u_{12}(1, hc^{\frac{\nu}{1-\nu}})}{u_1(1, hc^{\frac{\nu}{1-\nu}})} = (1-\nu)c\frac{u_{11}(c, h)}{u_1(c, h)} - \nu h\frac{u_{12}(c, h)}{u_1(c, h)},$$

which holds for all c and h. We conclude that the right-hand side of equation (A.2) only depends on $hc^{\frac{\nu}{1-\nu}}$, i.e., we can write

$$(1-\nu)c\frac{u_{11}(c, h)}{u_1(c, h)} - \nu h\frac{u_{12}(c, h)}{u_1(c, h)} = f_1(hc^{\frac{\nu}{1-\nu}}), \tag{A.3}$$

where f_1 is then defined by the expression on the left-hand side of equation (A.2) evaluated at $c\gamma^{1-\nu} = 1$. Differentiating (12) with respect to c gives

$$\frac{u_{12}(c, h)u_1(c, h) - u_{11}(c, h)u_2(c, h)}{u_1(c, h)^2} = \frac{c^{\frac{\nu}{1-\nu}}q(x)}{1-\nu} + \frac{\nu c^{\frac{1}{1-\nu}}v_1'(x)hc^{\frac{\nu}{1-\nu}-1}}{1-\nu} \equiv c^{\frac{\nu}{1-\nu}}f_2(x),$$

where we used the notation $x = hc^{\frac{\nu}{1-\nu}}$ and the last equality simply defines a new function f_2. Then, again using the characterization of the MRS function to replace

$\frac{u_2(c,h)}{u_1(c,h)} = c^{\frac{1}{1-\nu}} q(hc^{\frac{\nu}{1-\nu}})$, we obtain

$$\frac{u_{12}(c,h)}{u_1(c,h)} - \frac{u_{11}(c,h)}{u_1(c,h)} c^{\frac{1}{1-\nu}} q(x) = c^{\frac{\nu}{1-\nu}} f_2(x),$$

and hence

$$h\frac{u_{12}(c,h)}{u_1(c,h)} = \frac{u_{11}(c,h)}{u_1(c,h)} hc^{\frac{1}{1-\nu}} q(x) + hc^{\frac{\nu}{1-\nu}} f_2(x) = c\frac{u_{11}(c,h)}{u_1(c,h)} xq(x) + xf_2(x).$$

This expression can be combined with equation (A.3) to conclude that $\frac{-cu_{11}(c,h)}{u_1(c,h)}$ must be a function only of x; we call this function p.[26] ■

Proof of Theorem 1

Proof. We will now combine the information in Lemmata 1 and 2 to complete our proof of Theorem 1. We do this in two steps. First we analyze the case with $\nu \neq 0$ and then the case with $\nu = 0$. Note that the case with $\nu = 0$ is already discussed in King, Plosser and Rebelo (1988).

The strategy of the proof is very similar in the two cases. First, we integrate the RRA_c function in Lemma 2 with respect to c to obtain a functional form for u_1. As we integrate with respect to c, an unknown function of h appears. Then, by differentiating the obtained function for u_1 with respect to h we arrive at an expression that can be compared to a restriction on $\frac{u_{12}}{u_1}$ found in the proof of Lemma 2. This comparison gives us some additional restrictions on the unknown function of h. Thus, since the proof of Lemma 2 uses Lemma 1, we are in effect making sure that the functional form we arrive at is consistent with both our lemmata. Having arrived at a form for u_1, we again integrate to deliver a candidate for u. Due to the

[26] The function $p(x)$ is thus defined by

$$-(1-\nu)p(x) + \nu\left(p(x)xq(x) - xf_2(x)\right) = f_1(x),$$

which straightforwardly offers a solution (that will depend on q, f_1, and f_2).

integration a new unknown function of h again appears, but we can again restrict this function by differentiating our candidate u with respect to h and comparing the result to Lemma 1. This, then, delivers our final functional form.

Case with $\nu \neq 0$: note that the characterization of the $\mathrm{RRA_c}$ function in Lemma 2 can be restated as

$$\frac{\partial \log u_1(c,h)}{\partial \log(c)} = -p\left(\exp\left(\log(h) + \frac{\nu}{1-\nu}\log(c)\right)\right).$$

This equation can be integrated straightforwardly with respect to $\log(c)$ to arrive at

$$u_1(c,h) = f_3(hc^{\frac{\nu}{1-\nu}})m_1(h), \tag{A.4}$$

where f_3 is a new function of x and m_1 is an arbitrary function of h.[27]

Now observe that it follows from the proof of Lemma 2 that also $h\frac{u_{12}(c,h)}{u_1(c,h)}$ can be written as a function of x alone: it equals $-p(x)xq(x) + xf_2(x)$. We use this fact to further restrict the function m_1. In particular, by taking derivatives in equation (A.4) with respect to h, multiplying by h, and dividing by u_1, we obtain an expression for $h\frac{u_{12}(c,h)}{u_1(c,h)}$ that can be written as

$$f_4(hc^{\frac{\nu}{1-\nu}}) + \frac{m_1'(h)h}{m_1(h)},$$

where f_4 is defined by $f_4(x) \equiv f_3'(x)x/f_3(x)$. For the consistency of these two expressions for $h\frac{u_{12}(c,h)}{u_1(c,h)}$—the one just stated, and the arbitrary function of x given above ($-p(x)xq(x)+xf_2(x)$)—it must be that $\frac{m_1'(h)h}{m_1(h)}$ is a constant.[28] Hence, $m_1(h) =$

[27] The integration delivers an expression for $\log u_1(c,h)$ as a function of $\log x$ plus a function of h. The latter function can only be a function of h since c was integrated over. The function of $\log x$ can be rewritten as a function of x. Equation (A.4) is then obtained after raising e to the left- and right-hand sides of this equation and f_3 and m_1 are defined accordingly.

[28] If $\frac{m_1'(h)h}{m_1(h)}$ would depend on h, consistency could not be fulfilled for any given combination of c and h.

$A_1 h^\kappa$ for some constants A_1 and κ, i.e., it is isoelastic. Using this fact in (A.4) gives

$$u_1(c,h) = f_3(hc^{\frac{\nu}{1-\nu}})A_1 h^\kappa. \tag{A.5}$$

Since $\nu \neq 0$, the expression on the right-hand side can equivalently be written $f_5(h^{\frac{1-\nu}{\nu}}c)h^\kappa$, by defining $f_5(x) = A_1 f_3(x^{\frac{1-\nu}{\nu}})$. Therefore, (A.5) can be easily integrated with respect to c to deliver

$$u(c,h) = f_6(hc^{\frac{\nu}{1-\nu}})h^{\kappa-\frac{1-\nu}{\nu}} + m_2(h), \tag{A.6}$$

where f_6 is the new function that results from the integration of f_5 over c and m_2 is an arbitrary function of h (as the integration was over c). With the aim of further restricting m_2, we can express u_2 as

$$u_2(c,h) = u_1(c,h)c^{\frac{1}{1-\nu}}q(x) = f_3(x)A_1 h^\kappa c^{\frac{1}{1-\nu}}q(x) = f_7(hc^{\frac{\nu}{1-\nu}})h^{\kappa-\frac{1}{\nu}}, \tag{A.7}$$

where we have used the characterization of the MRS function in Lemma 1, (A.5), and finally the definition $f_7(x) \equiv f_3(x)A_1 x^{\frac{1}{\nu}}q(x)$. Thus, we can now check consistency by taking the derivative of u with respect to h in (A.6) and comparing with (A.7). The derivative becomes

$$\left(\kappa - \frac{1-\nu}{\nu}\right) f_6(x)h^{\kappa-\frac{1}{\nu}} + c^{\frac{\nu}{1-\nu}} f_6'(x)h^{\kappa-\frac{1-\nu}{\nu}} + m_2'(h) \equiv f_8(x)h^{\kappa-\frac{1}{\nu}} + m_2'(h),$$

where the equality comes from collecting terms and defining a new function f_8 accordingly. For consistency, thus, this expression has to equal $f_7(x)h^{\kappa-\frac{1}{\nu}}$ for all x and h. This is possible if and only if $m_2'(h) = A_2 h^{\kappa-\frac{1}{\nu}}$, where A_2 is a constant. Concentrating first on the case where $\kappa - \frac{1}{\nu} \neq -1$, we obtain $m_2(h) = \left(1+\kappa-\frac{1}{\nu}\right)^{-1} A_2 h^{1+\kappa-\frac{1}{\nu}} + A_3 \equiv A_4 h^{1+\kappa-\frac{1}{\nu}} + A_3$. The constant A_3 can be set arbitrarily as it does not affect choice. The second term in (A.6) can thus be merged together with the first term using factorization and we can write $u(c,h)$ as $f_9(x)h^{1+\kappa-\frac{1}{\nu}}+A_3$, with $f_9(x)\equiv f_6(x)+A_4$. Now note that $h^{1+\kappa-\frac{1}{\nu}}=x^{1+\kappa-\frac{1}{\nu}}c^{-\frac{\nu}{1-\nu}\left(1+\kappa-\frac{1}{\nu}\right)}$, so that $u(c,h)$ can be written as $f_9(x)x^{1+\kappa-\frac{1}{\nu}}c^{-\frac{\nu}{1-\nu}\left(1+\kappa-\frac{1}{\nu}\right)}+A_3$. Now define $v(x)\equiv\left((1-\sigma)f_9(x)x^{1+\kappa-\frac{1}{\nu}}\right)^{\frac{1}{1-\sigma}}$ and $\sigma\equiv\kappa\frac{\nu}{1-\nu}$ and we conclude that we can write $u(c,h)=\frac{(c\cdot v(x))^{1-\sigma}-1}{1-\sigma}$ (where A_3 has been set to $-1/(1-\sigma)$).

In the special case where $1+\kappa=1/\nu$, we obtain from equation (A.6) that $u(c,h)=f_6(hc^{\frac{\nu}{1-\nu}})+m_2(h)$, but we also see from the arguments above that $m_2(h)$ has to equal $A_2\log h+A_5$, where A_5 is again an arbitrary constant. Since (given $\nu\neq 0$) we can write $\log(h)=\log(x)-\frac{\nu}{1-\nu}\log(c)$, our candidate u can be rewritten as $u(c,h)=f_6(x)-A_2\frac{\nu}{1-\nu}\log(c)+A_2\log(x)+A_5$. The constant A_5 can be set to zero and we can write $u(c,h)=-A_2\frac{\nu}{1-\nu}\left[\log(c)-\frac{1-\nu}{A_2\nu}f_6(x)-\frac{1-\nu}{\nu}\log(x)\right]$. The factorized constant can be normalized to -1 (as it does not affect choice), and we can then define $\log v(x)\equiv f_6(x)+\frac{1-\nu}{\nu}\log(x)$, an arbitrary function; this concludes the case $1+\kappa=1/\nu$. Hence we obtain the utility function

$$u(c,h)=\begin{cases}\frac{\left(c\cdot v(hc^{\frac{\nu}{1-\nu}})\right)^{1-\sigma}-1}{1-\sigma} & \text{if } \sigma\neq 1\\ \log(c)+\log v(hc^{\frac{\nu}{1-\nu}}), & \text{if } \sigma=1.\end{cases}$$

Case with $\nu=0$: in this case we can rewrite the RRA_c function in Lemma 2 as

$$\frac{\partial\log u_1(c,h)}{\partial\log(c)}=-p(h). \tag{A.8}$$

We can integrate this equation with respect to $\log c$ to obtain

$$\log u_1(c,h)=-p(h)\log(c)+m_3(h), \tag{A.9}$$

where m_3 is an arbitrary function, given that we integrated over c. Differentiating with respect to h then gives

$$\frac{u_{12}(c,h)}{u_1(c,h)} = -p'(h)\log c + m_3'(h). \tag{A.10}$$

From the proof of Lemma 2 we know that $\frac{u_{12}(c,h)}{u_1(c,h)}$ must be possible to write as a function of h alone (recall that $\nu = 0$). From this we conclude that we must have $p'(h) = 0$, i.e., the only version of equation (A.9) that is possible is $\log u_1(c,h) = -\sigma\log(c) + m_3(h)$, where σ is a constant. Using this fact and raising e to both sides of (A.9) then delivers

$$u_1(c,h) = c^{-\sigma} m_4(h), \tag{A.11}$$

where $m_4(h) = \exp(m_3(h))$. Integrating (A.11) with respect to c we can write

$$u(c,h) = \begin{cases} \frac{(c\cdot v(h))^{1-\sigma}-1}{1-\sigma} + m_5(h) & \text{if } \sigma \neq 1 \\ m_4(h)\log(c) + \log v(h) & \text{if } \sigma = 1; \end{cases} \tag{A.12}$$

here, in the first equation $-1/(1-\sigma) + m_5$ is another function (of h) that appears because of the integration over c and $v(h)$ is defined from $\frac{v(h)^{1-\sigma}-1}{1-\sigma} = m_4(h)$, whereas in the second equation $\log v(h)$ is the function that appears due to the integration.

We will now, along the lines of the case where $\nu \neq 0$, show that m_4 and m_5 will have to have very specific forms. We look at each in turn. So in the case with $\sigma \neq 1$, combine (A.11) with Lemma 1 to write

$$u_2(c,h) = c^{1-\sigma} q(h) m_3(h). \tag{A.13}$$

This can be contrasted with the result of differentiating (A.12) with respect to h, an operation that yields

$$u_2(c,h) = c^{1-\sigma} v(h)^{-\sigma} v'(h) + m_5'(h).$$

Since these last two equations both have to hold for all c and h, it must be that $m_5'(h) = 0$, i.e., that $m_5(h)$ is a constant (which can be abstracted from).

Turning to the case where $\sigma = 1$, along the same lines we again derive two expressions for u_2 and check consistency. Combining (A.11) with Lemma 1 one obtains that u_2 cannot depend on c. Differentiating the second line of (A.12) with respect to h, however, delivers a function of c unless $m_4(h)$ is a constant; as it does not affect choice, we set this constant to 1.

This is our final characterization and we have now reproduced the statement in our main theorem. In summary, in the $\sigma \neq 1$ case we obtain $u(c,h) = \frac{(c \cdot v(h))^{1-\sigma}-1}{1-\sigma}$ and in the $\sigma = 1$ case we obtain $\log(c) + \log v(h)$. This completes the proof for the case $\nu = 0$. ■

Appendix B

B.1 A model with consumer heterogeneity

B.1.1 The Aiyagari model without growth

The consumer's problem: for all $(\omega, \epsilon, \beta)$,

$$V(\omega, \epsilon, \beta) = \max_{k', h} u(\omega + h\epsilon w - k', h) + \beta E[V(k'(1 - \delta + r), \epsilon', \beta') | \epsilon, \beta]$$

s.t. $k' \geq \underline{k}, h \in [0, \infty)$. This leads to decision rules $f^k(\omega, \epsilon, \beta)$ and $f^h(\omega, \epsilon, \beta)$.

Labor income is $\epsilon \in \{\epsilon_1, \epsilon_2, \ldots, \epsilon_I\}$ and $\beta \in \{\beta_1, \beta_2, \ldots, \beta_J\}$, with constant and exogenous first-order Markov transition probabilities $\pi(\epsilon', \beta' | \epsilon, \beta)$.

We assume that the economy produces with a neoclassical production function $F(\bar{k}, \bar{h})$ and the production factors earn their marginal products. **Stationary equilibrium:** prices r and w, a value function V, decision rules f^k and f^h, and a stationary distribution Γ such that

1. $f^k(\omega, \epsilon, \beta)$ and $f^h(\omega, \epsilon, \beta)$ attain the maximum in the consumer's problem for all $(\omega, \epsilon, \beta)$.

2. $r = F_1(\bar{k}, \bar{h})$ and $w = F_2(\bar{k}, \bar{h})$, where $\bar{k} \equiv (\sum_{\epsilon,\beta} \int_\omega \omega \Gamma(d\omega, \epsilon, \beta))/(1 - \delta + r)$ and $\bar{h} \equiv \sum_{\epsilon,\beta} \int_\omega \epsilon f^h(\omega, \epsilon, \beta) \Gamma(d\omega, \epsilon, \beta)$.

3. $\Gamma(B, \epsilon, \beta) = \sum_{\hat{\epsilon},\hat{\beta}} \pi_{\epsilon,\beta|\hat{\epsilon},\hat{\beta}} \int_{\omega: f^k(\omega,\hat{\epsilon},\hat{\beta}) \in B} \Gamma(d\omega, \hat{\epsilon}, \hat{\beta})$ for all Borel sets B and for all (ϵ, β).

B.1.2 The Aiyagari model with growth

The consumer's problem: for all $(\omega, \epsilon, \beta)$,

$$V_t(\omega, \epsilon, \beta) = \max_{k',h} u(\omega + h\epsilon w_t - k', h) + \beta E[V_{t+1}(k'(1-\delta+r), \epsilon', \beta')|\epsilon, \beta]$$

s.t. $k' \geq \underline{k} g^{t+1}, h \in [0, \infty)$. Notice, here, that the borrowing constraint changes over time (unless $\underline{k} = 0$) and gets less and less stringent with $\underline{k} < 0$. This leads to decision rules $f_t^k(\omega, \epsilon, \beta)$ and $f_t^h(\omega, \epsilon, \beta)$.

Labor income and discount factors are as before. Now, however, note that $w_t = \gamma^t w$ for all t.

A balanced-growth equilibrium: growth rates g and g_h, prices r and w_t, a value function V_t, decision rules f_t^k and f_t^h, and distributions Γ_t such that, for all t,

1. $g = \gamma g_h$.

2. $f_t^k(\omega, \epsilon, \beta)$ and $f_t^h(\omega, \epsilon, \beta)$ attain the maximum in the consumer's problem for all $(\omega, \epsilon, \beta)$.

3. $r = F_1(\bar{k}_t, \gamma^t \bar{h}_t)$ and $w_t = \gamma^t F_2(\bar{k}_t, \gamma^t \bar{h}_t)$, where $\bar{k}_t \equiv (\sum_{\epsilon,\beta} \int_\omega \omega \Gamma_t(d\omega, \epsilon, \beta))/(1-\delta + r)$ and $\bar{h}_t \equiv \sum_{\epsilon,\beta} \int_\omega \epsilon f_t^h(\omega, \epsilon, \beta) \Gamma_t(d\omega, \epsilon, \beta)$.

4. $\Gamma_{t+1}(B, \epsilon, \beta) = \sum_{\hat{\epsilon},\hat{\beta}} \pi_{\epsilon,\beta|\hat{\epsilon},\hat{\beta}} \int_{\omega: f_t^k(\omega,\hat{\epsilon},\hat{\beta}) \in B} \Gamma_t(d\omega, \hat{\epsilon}, \hat{\beta})$ for all Borel sets B and for all (ϵ, β).

5. $f_t^k(\omega g^t, \epsilon, \beta) = g^t f_0^k(\omega, \epsilon, \beta)$, $f_t^h(\omega g^t, \epsilon, \beta) = g_h^t f_0^h(\omega, \epsilon, \beta)$, and $\Gamma_t(B g^t, \epsilon, \beta) = \Gamma_0(B, \epsilon, \beta)$ for all ω, B, and (ϵ, β).

Note that due to growth, the distribution over ω will not be stationary. However, as we will show below, once ω is detrended by the appropriate growth rate we obtain a stationary distribution.

B.1.3 Transforming the Aiyagari model with growth

Using the last condition of the balanced-growth equilibrium, note that in the third condition we can write $\bar{k}_t = (\sum_{\epsilon,\beta}\int_\omega \omega\Gamma_0(\frac{d\omega}{g^t},\epsilon,\beta))/(1-\delta+r)$, which is equivalent to $\tilde{\bar{k}}_t \equiv \frac{\bar{k}_t}{g^t} = (\sum_{\epsilon,\beta}\int_{\tilde{\omega}} \tilde{\omega}\Gamma_0(d\tilde{\omega},\epsilon,\beta))/(1-\delta+r)$, where we have defined $\tilde{\omega} = \omega/g^t$. Notice also that $\tilde{\bar{k}}_t = \tilde{\bar{k}}$, i.e., a constant, in a balanced-growth equilibrium.

Similarly, we obtain $\bar{h}_t = \sum_{\epsilon,\beta}\int_\omega \epsilon g_h^t f_0^h(\frac{\omega}{g^t},\epsilon,\beta)\Gamma_0(\frac{d\omega}{g^t},\epsilon,\beta)$, implying that $\tilde{\bar{h}}_t \equiv \frac{\bar{h}_t}{g_h^t} = \sum_{\epsilon,\beta}\int_{\tilde{\omega}} \epsilon f_0^h(\tilde{\omega},\epsilon,\beta)\Gamma_0(d\tilde{\omega},\epsilon,\beta)$, which also is constant under balanced growth: $\tilde{\bar{h}}_t = \tilde{\bar{h}}$.

Given $g = \gamma g_h$ and that F_1 and F_2 are both homogeneous of degree 0, we now see that the two firm first-order conditions can be expressed as

$$r = F_1(\tilde{\bar{k}},\tilde{\bar{h}}) \quad \text{and} \quad w_0 = F_2(\tilde{\bar{k}},\tilde{\bar{h}}). \tag{B.1}$$

Turning to the fourth equilibrium condition, using the (very) last condition stating that the distribution is (in an appropriate sense) constant on the balanced growth path, we obtain

$$\Gamma_0(B/g^{t+1},\epsilon,\beta) = \sum_{\hat{\epsilon},\hat{\beta}} \pi_{\epsilon,\beta|\hat{\epsilon},\hat{\beta}} \int_{\tilde{\omega}: f_0^k(\tilde{\omega},\hat{\epsilon},\hat{\beta})g^t \in B} \Gamma_0(d\tilde{\omega},\hat{\epsilon},\hat{\beta}),$$

where we used the definition of $\tilde{\omega}$. Defining $\tilde{B} = B/g^t$ for any Borel set B, we obtain

$$\Gamma_0(\tilde{B}/g,\epsilon,\beta) = \sum_{\hat{\epsilon},\hat{\beta}} \pi_{\epsilon,\beta|\hat{\epsilon},\hat{\beta}} \int_{\tilde{\omega}: f_0^k(\tilde{\omega},\hat{\epsilon},\hat{\beta}) \in \tilde{B}} \Gamma_0(d\tilde{\omega},\hat{\epsilon},\hat{\beta}). \tag{B.2}$$

Looking at consumer optimization under balanced growth, finally, we obtain (after using the same kinds of definitions as above),

$$V_t(\tilde{\omega}g^t,\epsilon,\beta) = \max_{\tilde{k}',\tilde{h}} u(\tilde{\omega}g^t + \tilde{h}g_h^t\epsilon w_0\gamma^t - \tilde{k}'g^{t+1}, \tilde{h}g_h^t) + \beta E[V_{t+1}(\tilde{k}'g^{t+1}(1-\delta+r),\epsilon',\beta')|\epsilon,\beta]$$

s.t. $\tilde{k}' g^{t+1} \geq \underline{k} g^{t+1}, \tilde{h} g_h^t \in [0, \infty)$.

Now consider our instantaneous utility function of Theorem 1 for u and let $g_h = \gamma^{-\nu}$ and $g = \gamma^{1-\nu}$. Then $g^{t(1-\sigma)}$ can be factorized out from u. Dividing both sides of the equation by this quantity and defining $V_t(\tilde{\omega} g^t, \epsilon, \beta) \equiv g^{t(1-\sigma)} \tilde{V}(\tilde{\omega}, \epsilon, \beta)$, we can write

$$\tilde{V}(\tilde{\omega}, \epsilon, \beta) = \max_{\tilde{k}', \tilde{h}} u(\tilde{\omega} + \tilde{h} \epsilon w_0 - \tilde{k}' g, \tilde{h}) + \beta g^{1-\sigma} E[\tilde{V}(\tilde{k}'(1 - \delta + r), \epsilon', \beta') | \epsilon, \beta] \quad \text{(B.3)}$$

s.t. $\tilde{k}' \geq \underline{k}, \tilde{h} \in [0, \infty)$, with associated policy functions $\tilde{f}_t^k(\tilde{\omega}, \epsilon, \beta)$ and $\tilde{f}_t^h(\tilde{\omega}, \epsilon, \beta)$.

Now r, w_0, $\tilde{V}$, $\tilde{f}^k$, $\tilde{f}^h$, and Γ_0, determined by equations (B.1), (B.2), and (B.3), define a stationary equilibrium. Three items differ compared to the formulation above for the stationary equilibrium without growth: the discount factors in the consumer's problem are all multiplied by $g^{1-\sigma}$, an additional gross "cost" of saving, g, appears, and g also appears in the argument on the left-hand side of the equation determining the stationary distribution.

B.2 Closed-form solutions for specific cases

B.2.1 Unitary elasticity and a constant x_t

To illustrate transitional dynamics in hours worked, suppose we have $v(x) = 1 - \phi h c^{\frac{\nu}{1-\nu}}$, with $\phi > 0$ and $1/2 > \nu \geq 0$, which gives the following utility function.[29]

$$u(c, h) = \log(c) + \log\left(1 - \phi h c^{\frac{\nu}{1-\nu}}\right). \quad \text{(B.4)}$$

[29] Note that with $\nu = 0$ and $\phi = 1$ we obtain the symmetric Cobb-Douglas function, which belongs to the KPR class.

Under the assumed parameter restrictions, the function (B.4) is increasing in c, decreasing in h, and concave if $1-2\nu > \phi(1-\nu)x$, where $x \equiv hc^{\frac{\nu}{1-\nu}}$.[30] Under Cobb-Douglas technology and 100% depreciation, (19) can explicitly be solved to obtain $\bar{x} = \frac{(1-\alpha)(1-\nu)\frac{1}{\phi}}{1-\alpha+(1-\alpha\beta\eta)(1-\nu)}$. With $x_t = \bar{x}$, $\beta\eta < 1$, and $\nu < 1/2$, concavity is ensured since $(1-2\nu)(1-\alpha)+(1-\nu)(1-\alpha\beta\eta) > (1-\alpha)(1-\nu)^2$. We then obtain the following closed-form solution for the detrended variables $\hat{h}_t \equiv h_t\gamma^{\nu t}$ and $\hat{c}_t \equiv c_t\gamma^{-(1-\nu)t}$.

$$\hat{h}_t = (1-\alpha\beta\eta)^{\frac{-\nu}{1-\alpha\nu}} \left[\frac{(1-\alpha)(1-\nu)\frac{1}{\phi}}{(1-\alpha)+(1-\alpha\beta\eta)(1-\nu)}\right]^{\frac{1-\nu}{1-\alpha\nu}} \hat{k}_t^{\frac{-\nu\alpha}{1-\nu\alpha}} \tag{B.5}$$

and

$$\hat{c}_t = \hat{k}_t^{\frac{\alpha(1-\nu)}{1-\nu\alpha}} (1-\alpha\beta\eta)^{\frac{1-\nu}{1-\alpha\nu}} \left[\frac{(1-\alpha)(1-\nu)\frac{1}{\phi}}{1-\alpha+(1-\alpha\beta\eta)(1-\nu)}\right]^{\frac{(1-\nu)(1-\alpha)}{1-\alpha\nu}}. \tag{B.6}$$

For the law of motion of the capital stock we obtain

$$\hat{k}_{t+1} = \hat{k}_t^{\frac{\alpha(1-\nu)}{1-\alpha\nu}} \left(\frac{\alpha\beta}{\gamma^{1-\nu}}\right) (1-\alpha\beta\eta)^{-\nu\frac{(1-\alpha)}{1-\alpha\nu}} \left[\frac{(1-\alpha)(1-\nu)\frac{1}{\phi}}{1-\alpha+(1-\alpha\beta\eta)(1-\nu)}\right]^{\frac{(1-\nu)(1-\alpha)}{1-\alpha\nu}}. \tag{B.7}$$

Along the balanced growth path we have

$$\hat{k}^\star = \left(\frac{\alpha\beta}{\gamma^{1-\nu}}\right)^{\frac{1-\nu\alpha}{1-\alpha}} (1-\alpha\beta\eta)^{-\nu} \left[\frac{(1-\alpha)(1-\nu)\frac{1}{\phi}}{1-\alpha+(1-\alpha\beta\eta)(1-\nu)}\right]^{1-\nu}, \tag{B.8}$$

---

[30] To see this, note that we have $u_1(c,h) = \frac{1}{c(1-\nu)}\left[\frac{1-\nu-x\phi}{1-x\phi}\right]$, which is strictly positive since $(1-\nu) \geq \frac{1-2\nu}{1-\nu} > x\phi$. Moreover, we have $u_2(c,h) = -\frac{1}{h}\left[\frac{x\phi}{1-x\phi}\right] < 0$. The second and cross-derivatives are $u_{11}(c,h) = -\frac{1}{c^2(1-\nu)^2}\left[1-\nu-\frac{\nu(1-\nu-x\phi)}{(1-x\phi)^2}\right]$, $u_{22}(c,h) = -\frac{1}{h^2}\left[\frac{x\phi}{1-x\phi}\right]^2 < 0$, and $u_{12}(c,h) = -\frac{1}{ch(1-\nu)}\frac{\nu x\phi}{(1-x\phi)^2} \leq 0$, so that the Hessian becomes $u_{11}(c,h)u_{22}(c,h) - u_{12}(c,h)^2 = \frac{\phi^2[(1-2\nu)-(1-\nu)x\phi]}{c^{2\frac{1-2\nu}{1-\nu}}(1-\nu)^2(1-x\phi)^3}$, which is strictly positive for $1-2\nu > (1-\nu)x\phi$.

$$\hat{h}^{\star} = \left(\frac{\alpha\beta}{\gamma^{1-\nu}}\right)^{\frac{-\nu\alpha}{1-\alpha}} (1-\alpha\beta\eta)^{-\nu} \left[\frac{(1-\alpha)(1-\nu)\frac{1}{\phi}}{1-\alpha+(1-\alpha\beta\eta)(1-\nu)}\right]^{1-\nu}, \tag{B.9}$$

and

$$\hat{c}^{\star} = \left(\frac{\alpha\beta}{\gamma^{1-\nu}}\right)^{\frac{\alpha(1-\nu)}{1-\alpha}} \left[\frac{(1-\alpha\beta\eta)(1-\alpha)(1-\nu)\frac{1}{\phi}}{1-\alpha+(1-\alpha\beta\eta)(1-\nu)}\right]^{1-\nu}. \tag{B.10}$$

B.2.2 CES production and no transitional dynamics in hours

Suppose preferences are as in (15) and the production function is

$$F\left[K_t, \gamma^t h_t \eta^t\right] = \left[\alpha K_t^{\frac{\varepsilon-1}{\varepsilon}} + (1-\alpha)\left(\gamma^t h_t \eta^t\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}. \tag{B.11}$$

With 100% depreciation, the transformed resource constraint reads

$$\eta\gamma^{1-\nu}\hat{k}_{t+1} = \left[\alpha\hat{k}_t^{\frac{\varepsilon-1}{\varepsilon}} + (1-\alpha)\hat{h}_t^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} - \hat{c}_t,$$

and the first-order conditions read

$$\left(\frac{\hat{c}_{t+1}}{\hat{c}_t}\right)^{\sigma} = \frac{\beta\alpha\hat{k}_{t+1}^{-\frac{1}{\varepsilon}}\left[\alpha\hat{k}_{t+1}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\alpha)\hat{h}_{t+1}^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{1}{\varepsilon-1}}}{\gamma^{\sigma(1-\nu)}}$$

and

$$(1-\alpha)\hat{h}_t^{-\frac{1}{\varepsilon}}\left[\alpha\hat{k}_t^{\frac{\varepsilon-1}{\varepsilon}} + (1-\alpha)\hat{h}_t^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{1}{\varepsilon-1}} \hat{c}_t^{-\sigma} = \psi\hat{h}_t^{\frac{1}{\theta}}, \tag{B.12}$$

where $\nu = \frac{\sigma-1}{\sigma+1/\theta}$. When we further impose the restriction $\sigma/\varepsilon = 1$, it is easy to guess and verify that these conditions are all fulfilled if

$$\hat{c}_t = \left(1-\eta(\alpha\beta)^{\frac{1}{\sigma}}\right)\left[\alpha\hat{k}_t^{1-\sigma} + (1-\alpha)\hat{h}_t^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{B.13}$$

and

$$\hat{h}_t = \left(\frac{1-\alpha}{\psi \left(1 - \eta(\alpha\beta)^{\frac{1}{\sigma}}\right)^{\sigma}} \right)^{\frac{\theta}{1+\sigma\theta}} = \hat{h}^*. \tag{B.14}$$

For the balanced-growth capital stock we then obtain

$$\hat{k}^* = \frac{(1-\alpha)^{\frac{1}{1-\sigma}} (\alpha\beta)^{\frac{1}{\sigma}}}{\left[\gamma^{\frac{(1-\sigma)(1+\theta)}{\theta\sigma+1}} - \alpha(\alpha\beta)^{\frac{1-\sigma}{\sigma}}\right]^{\frac{1}{1-\sigma}}} \left(\frac{1-\alpha}{\psi \left(1 - \eta(\alpha\beta)^{\frac{1}{\sigma}}\right)^{\sigma}} \right)^{\frac{\theta}{1+\sigma\theta}}. \tag{B.15}$$

Additional references in the appendices

Piketty, T., and E. Saez (2003), "Income Inequality in the United States, 1913–1998", *Quarterly Journal of Economics*, **118**(1), 1–41.

Piketty, T. (2014), *Capital in the twenty-first century*, Harvard University Press.

Appendix C

C.1 Additional figures

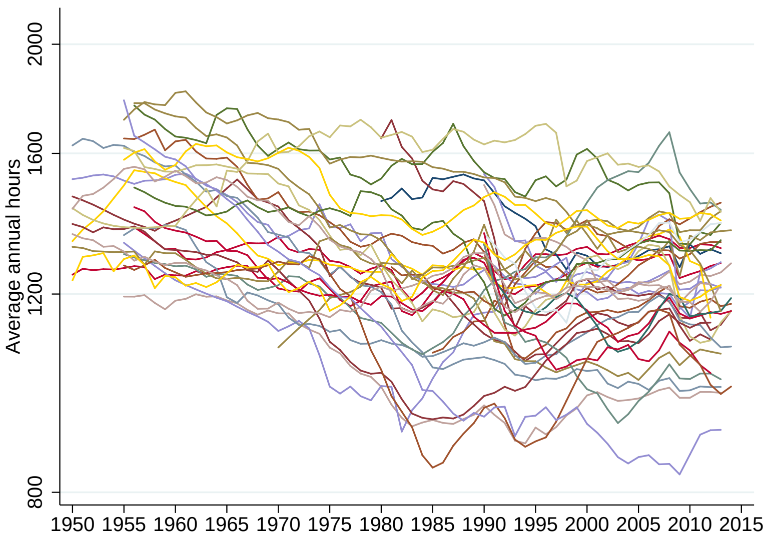


Figure C.1: **Average annual hours per capita aged 15–64, 1950–2015**

Notes: Source: GGDC Total Economy Database for total hours worked and OECD for the data on population aged 15–64. The figure is comparable to the ones in Rogerson (2006). The sample includes 37 countries. Regressing the logarithm of hours worked on time and country fixed effects gives a slope coefficient of -0.00336. The R^2 of the regression is 0.64.

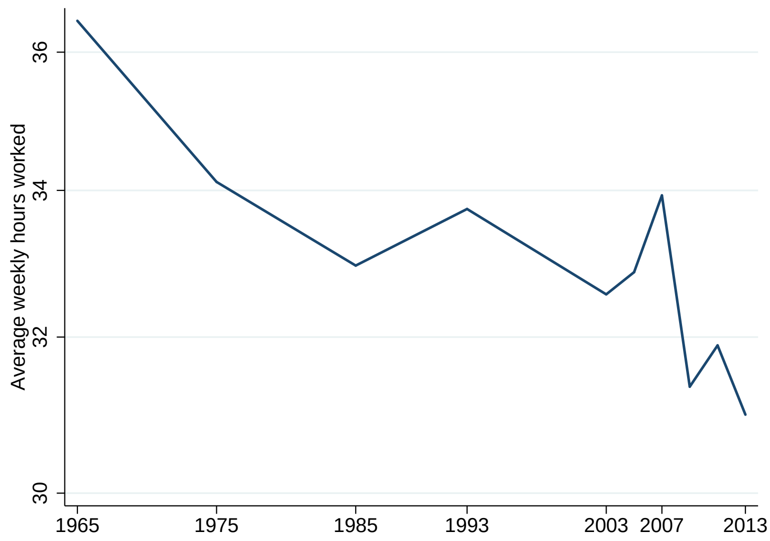


Figure C.2: **U.S. time used survey: Weekly hours worked**

Notes: Source: ATUS, following the methodology in Aguiar and Hurst (2007). The sample contains all non-retired, non-student individuals at age 21–65. For the years 1965–2003 the series is comparable to Aguiar and Hurst (2007) Table II and is updated until 2013 using the same methodology. Regressing the logarithm of hours worked on time gives a slope coefficient of -0.0024.

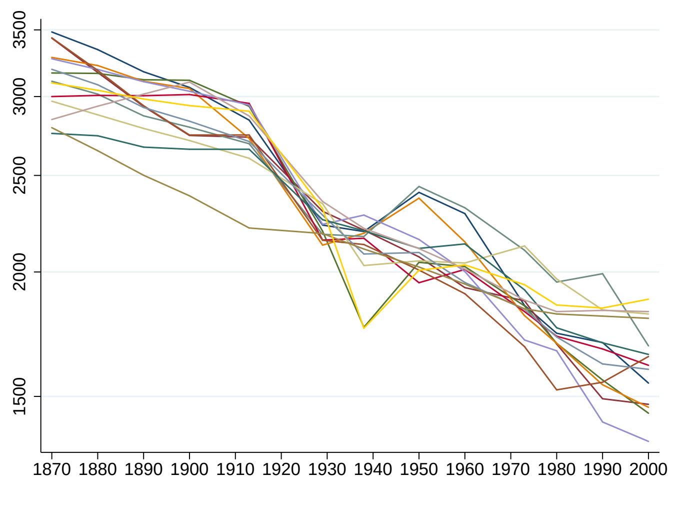


Figure C.3: **Hours worked per worker**

Notes: The figure shows data for the following countries: Belgium, Denmark, France, Germany, Ireland, Italy, Netherlands, Spain, Sweden, Switzerland, the U.K., Australia, Canada, and the U.S. The scale is logarithmic which suggests that hours fall at roughly 0.57 percent per year. Source: Huberman and Minns (2007).

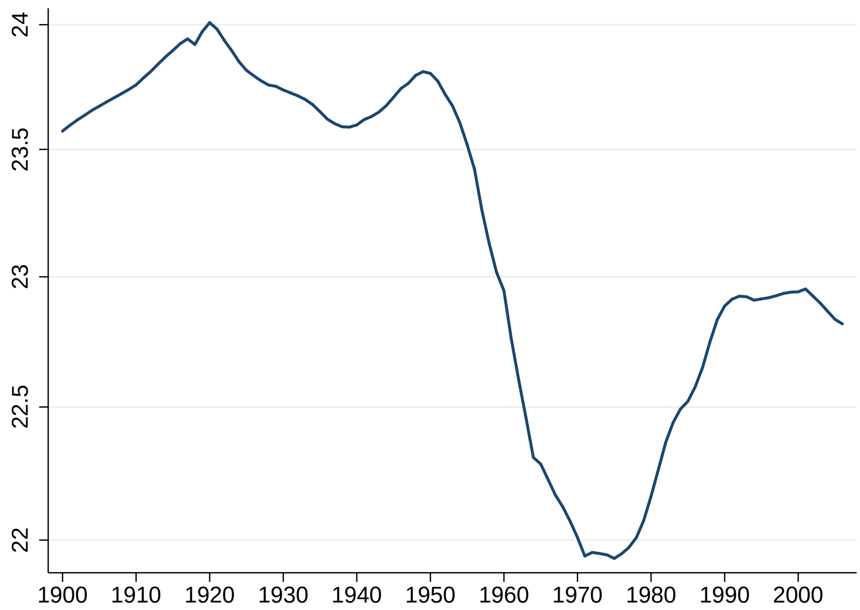
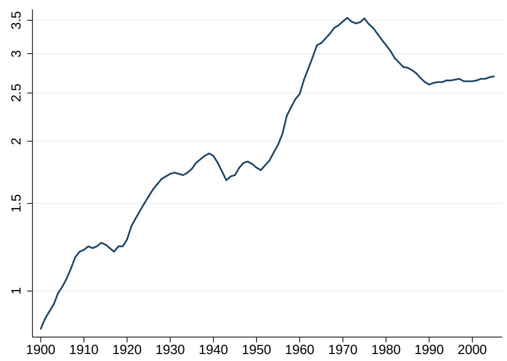
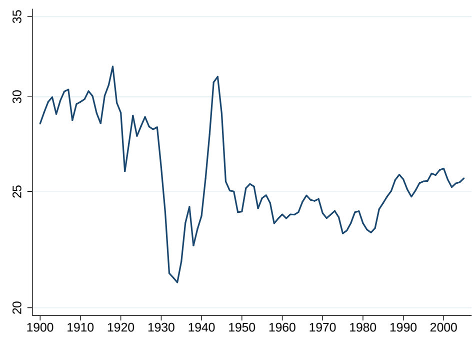


Figure C.4: **Effect of demographics on hours worked, U.S. 1900–2005**

Notes: The figure shows the implied average weekly hours worked per person aged 14+ over time by the variable demographical composition of the society. The scale is logarithmic. Hours worked of each age bracket are held constant at their value in 2005 and only the demographical composition is changing over time. The considered age brackets are 14–17; 18–24; 25–54; 55–64; and 65+. The figure looks very similar for other baseline years than 2005. Source: Ramey and Francis (2009) and U.S. Census.



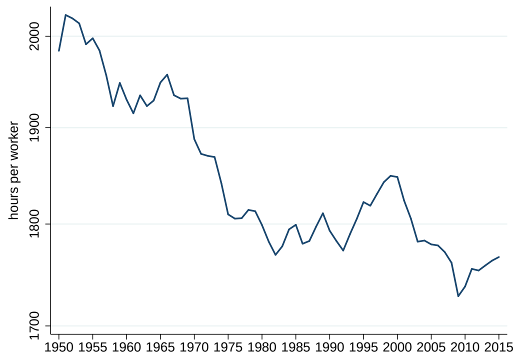
(a) Weekly hours of school



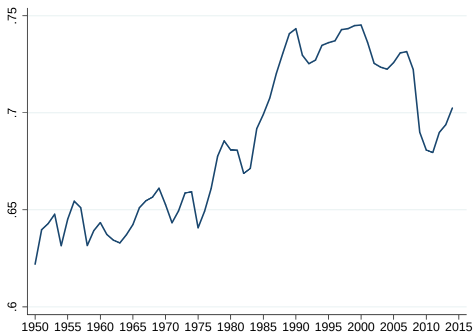
(b) Hours of work plus school

Figure C.5: Hours of schooling

Notes: The figure on the left shows average weekly hours of schooling (time in class and homework) per population aged 14+. The figure on the right shows average hours spent for work plus schooling per population aged 14+. The scale is logarithmic in both figures. Regressing the logarithm of hours worked plus schooling on time gives a slope coefficient of -0.0018. Source: Ramey and Francis (2009).



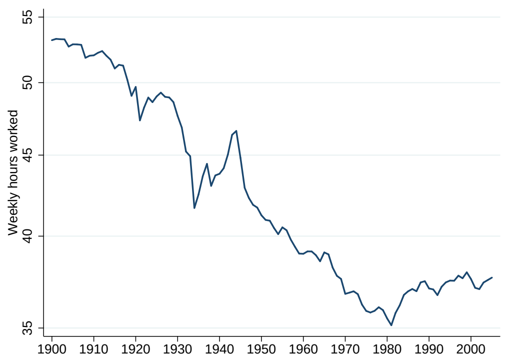
(a) Hours per worker



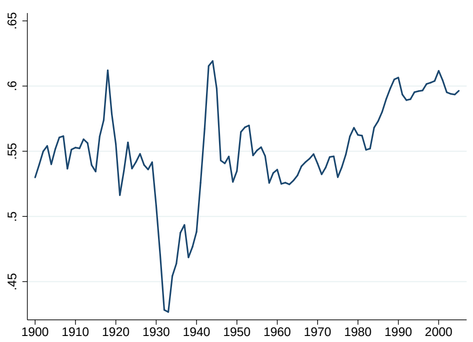
(b) Participation rate

Figure C.6: Hours per worker and participation rate in the postwar U.S.

Notes: The scale is logarithmic in the figure on hours worked per worker. Regressing the logarithm of hours worked per worker on time gives a slope coefficient of -0.002. Source: GGDC Total Economy Database for total hours worked and labor productivity and OECD for the data on population aged 15–64.



(a) Hours per worker



(b) Participation rate

Figure C.7: Hours per worker and participation rate in the U.S.

Notes: The scale is logarithmic in the figure on hours worked per worker. Regressing the logarithm of hours worked per worker on time gives a slope coefficient of -0.00418. Source: Ramey and Francis (2009).

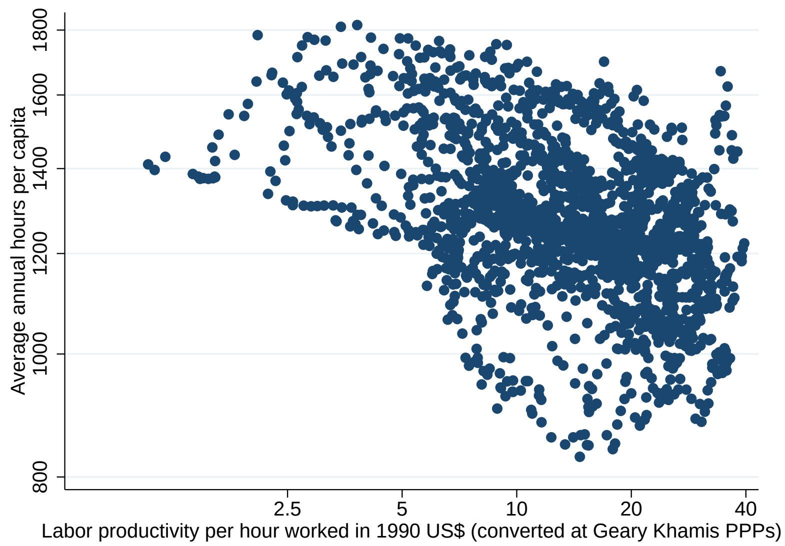


Figure C.8: **Hours worked vs. labor productivity**

Source: Source: GGDC Total Economy Database for total hours worked and labor productivity and OECD for the data on population aged 15–64. Regressing the logarithm of hours worked on the logarithm of labor productivity and a country fixed effect gives a slope coefficient of -0.13 and an R^2 of 0.69.

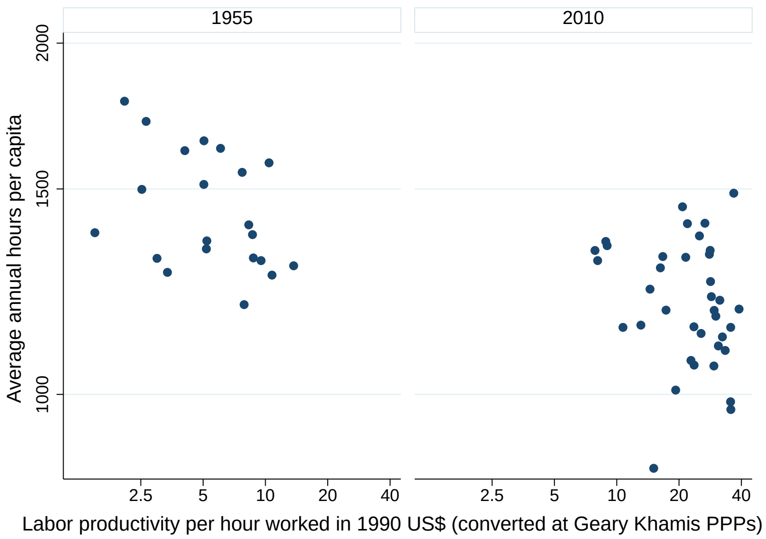


Figure C.9: **Hours worked vs. labor productivity**

Source: GGDC Total Economy Database for total hours worked and labor productivity and OECD for the data on population aged 15–64. The figure shows the scatter plot between labor productivity and hours worked for the years 1955 and 2010.

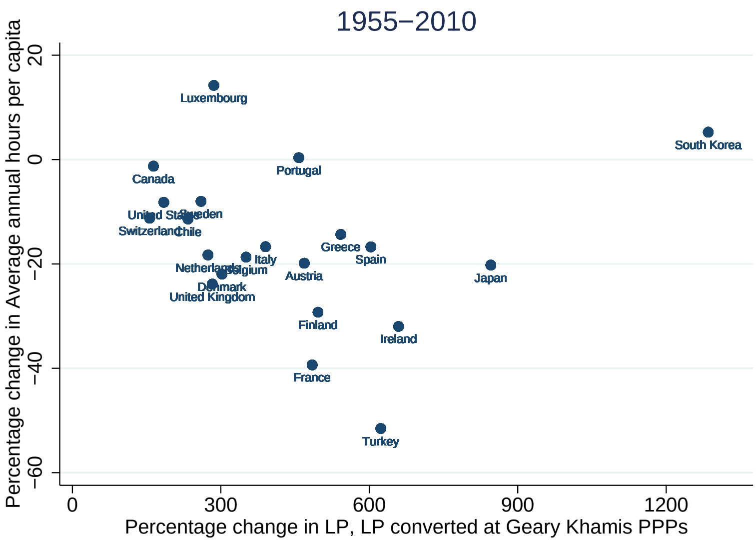
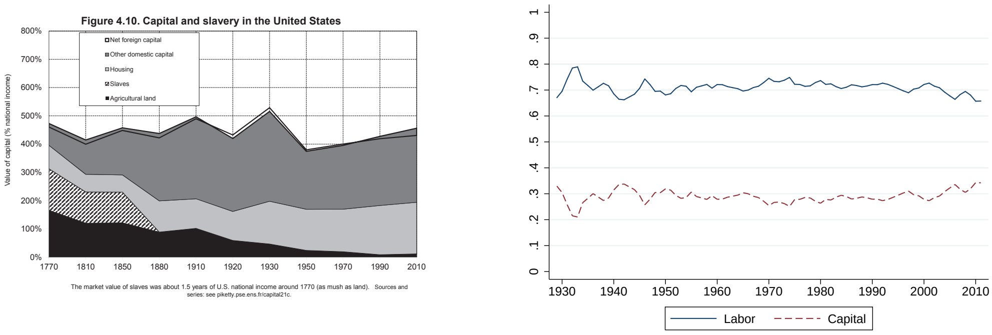


Figure C.10: **Changes in hours worked vs. labor productivity**

Source: GGDC Total Economy Database for total hours worked and labor productivity and OECD for the data on population aged 15–64.



(a) Capital-output ratio (b) Factor income shares

Figure C.11: Additional balanced-growth facts

Source: BEA; Piketty (2014) and updated series of Piketty and Saez (2003).